ON DIRECT PRODUCTS OF GENERALIZED SOLVABLE GROUPS

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Let G_α $(\alpha\in\Gamma)$ be a set of groups. The direct product $\prod\{G_\alpha|\alpha\in\Gamma\}$ is the set of all functions f on Γ such that $f(\alpha)\in G_\alpha$ for all $\alpha\in\Gamma$, with multiplication of functions defined componentwise. The direct sum $\sum\{G_\alpha|\alpha\in\Gamma\}$ is the subgroup of $\prod\{G_\alpha|\alpha\in\Gamma\}$ consisting of all functions f with finite support.

A collection $\mathfrak{B}$ of groups is called a class of groups if $E\in\mathfrak{B}$, and isomorphic images of $\mathfrak{B}$ groups are $\mathfrak{B}$ groups. We use the following notation of P. Hall [1]. If $\mathfrak{B}$ is a class of groups, $S(\mathfrak{B})$, $Q(\mathfrak{B})$, $DS(\mathfrak{B})$, $DP(\mathfrak{B})$ denote respectively the classes of groups which are subgroups, quotient groups, direct sums and direct products of $\mathfrak{B}$ groups.

The following theorem was proved by Merzulakov in [2].

THEOREM 1. *If $\mathfrak{B}$ is a class of groups satisfying*

(a) $S(\mathfrak{B})=\mathfrak{B}$,

(b) $Q(\mathfrak{B})=\mathfrak{B}$,

(c) *G is a finite $\mathfrak{B}$ group if and only if G is nilpotent, then $DP(\mathfrak{B})\neq\mathfrak{B}$.*

In this paper, a similar theorem is obtained for generalized solvable groups. Before stating these results, we need several definitions.

DEFINITION 1. Let G be a group, $x\in G$, $g\in G$. Define $[g, 0x]=g$, and inductively $[g, nx]=[[g, (n-1)x], x]$ for each positive integer n. x is called a left G Engel element if for each $g\in G$ there exists an integer $n=n(g)$ such that $[g, nx]=e$.

The Hirsch-Plotkin radical of a group G is the maximum normal locally nilpotent subgroup of G. We denote the Hirsch-Plotkin radical of G by $\phi_1(G)$.

DEFINITION 2. Let G be a group and $\phi_0(G)=E$. If α is not a limit ordinal, define $\phi_\alpha(G)$ by $\phi_\alpha(G)/\phi_{\alpha-1}(G)=\phi_1(G/\phi_{\alpha-1}(G))$. If α is a limit ordinal, define $\phi_\alpha(G)$ by $\phi_\alpha(G)=\bigcup\{\phi_\beta|\beta<\alpha\}$. If for some ordinal σ, $\phi_\sigma(G)=G$, G is called an LN-radical group.

In the following, $\mathfrak{L}$ will denote the class of LN-radical groups. If $G\in\mathfrak{L}$, and σ is the least ordinal for which $\phi_\sigma(G)=G$, σ is called the radical class of G. It is well known that $S(\mathfrak{L})=\mathfrak{L}$, $Q(\mathfrak{L})=\mathfrak{L}$, and that every solvable group is in $\mathfrak{L}$ [3]. It is easily shown that if n is a positive integer, there exist finite solvable groups of radical class n [4, p. 220].

We need the following theorem of Plotkin [3].