THE THIRD COHOMOLOGY GROUP OF A RING AND THE COMMUTATIVE COHOMOLOGY THEORY

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The cohomology groups of a ring depend not only on the ring but on a choice of category of which the ring is a member. In [4] it was shown that under very weak conditions on the category one could define the third cohomology group $\mathcal{E}^3(A, M)$ of a ring A with coefficients in a bimodule M as certain equivalence classes of exact sequences

$$0 \to M \to N \xrightarrow{\rho} B \to A \to 0. \tag{1}$$

The groups $\mathcal{E}^1(A, M)$ and $\mathcal{E}^2(A, M)$ were the derivations of A into M and extensions of A by M, respectively. We show here that if $\mathfrak{a}$ is an ideal of A and if M is an $A/\mathfrak{a}$ module, then there is an exact sequence

$$\begin{aligned} &0 \to \mathcal{E}^1(A/\mathfrak{a}, M) \xrightarrow{j_1^*} \mathcal{E}^1(A, M) \xrightarrow{i_1^*} \mathrm{Hom}_A(\mathfrak{a}, M) \xrightarrow{\Delta_1} \\ &\mathcal{E}^2(A/\mathfrak{a}, M) \xrightarrow{j_2^*} \mathcal{E}^2(A, M) \xrightarrow{i_2^*} \mathcal{C} \xrightarrow{\Delta_2} \mathcal{E}^3(A/\mathfrak{a}, M) \xrightarrow{j_3^*} \mathcal{E}^3(A, M), \end{aligned} \tag{2}$$

where $\mathcal{C}$ is an explicitly described submodule of $\mathrm{Ext}_A^1(\mathfrak{a}, M)$. (Cf. Harrison [5, Theorem 2].) We then show that for the category of commutative associative algebras over a coefficient field k, the group $\mathcal{E}^3(A, M)$ as defined in [4] coincides with that defined by Harrison in [5]. (An example of Barr in a note to appear [1] shows that in the category of commutative associative algebras, $\mathcal{E}^3(A, M)$ is not the first derived functor of the Baer group $\mathcal{E}^2(A, M)$ when the latter is considered as a functor of the module M.) More generally, two cohomology theories for a category of algebras or groups with sufficiently many projectives coincides if (i) each possesses an exact sequence analogous to (2) with $\mathcal{E}^1(A, M)$ the derivations of A into M, and (ii) $\mathcal{E}^n(A, M) = 0$ whenever A is projective.

In order to be brief, we prove the exactness of (2) explicitly only for commutative associative algebras over k, but the reader of [4] will observe that the considerations apply to any "category of interest" in the sense of that paper.

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