DIFFERENTIAL EQUATIONS OF THE MATRIX ELEMENTS OF THE DEGENERATE SERIES

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1. **Introduction.** Let G be a noncompact, semisimple Lie group, and let $U(G)$ be the universal enveloping algebra of its Lie algebra. [For notations used here, see [3].] Consider an irreducible representation ρ of G on a Hilbert space H. Let $I(G)$ be the central elements in $U(G)$, i.e., the Casimir operators of the Lie algebra. Then, for $\Delta \in I(G)$, $\rho(\Delta)$ is a multiple of the identity in H, which implies that the matrix elements of the representation, considered as functions on G, satisfy certain invariant differential equations, whose nonconstant terms are just the differential operators on G resulting from considering the elements of $I(G)$ as differential operators on G.

Following a suggestion of A. Koranyi, we now ask the following question. Suppose Δ is an element of $U(G)$ that is not in the center. Are there representations ρ which have the property that $\rho(\Delta)$ is a multiple of the identity also, hence that the matrix elements of ρ, considered as functions on G satisfy additional differential equations derived from Δ? A remark of this type might be useful in interpreting the role that the various types of representations play in the Plancherel formula for functions on the group, and in characterizing by differential equations the classes of harmonic functions on G/K (K = maximal compact subgroup) arising from various Poisson integral formulae.

In this note, we indicate how to find such Δ's for certain induced representations of G. Roughly, we show that the higher the degree of degeneracy of the representation, the more likely one is to find such operators. We use a differential geometric method already given in [2] and [3] to prove the vanishing of certain differential operators.

2. **Conditions for the vanishing of differential operators.** Consider a space with variables $(x_i) = x$ (choose the range of indices $1 \leqq i, j, \cdots, \leqq n$). Let Δ and X be linear differential operators on this space, of order r and 1 respectively. Then

$$\Delta = \sum A_{i_1 \cdots i_r} \frac{\partial^r}{\partial x_{i_1} \cdots \partial x_{i_r}} + (\cdots)$$

where $A_{i_1 \cdots i_r}(x)$ are functions on this space, depending symmetrically on the indices. [The terms $(\cdots)$ are of lower order.]