BOUNDS FOR THE FUNDAMENTAL SOLUTION OF A PARABOLIC EQUATION[1]

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1. **Introduction.** It is well known that the parabolic equation

$$(1) \qquad a_{ij}(x, t)u_{x_ix_j} + a_i(x, t)u_{x_i} + a(x, t)u - u_t = 0$$

possesses a fundamental solution provided that the coefficients are Hölder continuous. Here $x = (x_1, \cdots, x_n)$ denotes a point in E^n with $n \geqq 1$, t denotes a point on the real line, and we employ the convention of summation over repeated indices. The fundamental solution $g(x, t; \xi, \tau)$ can be constructed by the classical parametrix method, and it satisfies the inequality $0 \leqq g \leqq K\gamma$, where γ is the fundamental solution of $\alpha\Delta u = u_t$ for some constant $\alpha > 0$ and $K > 0$ is a constant which depends upon the Hölder norms of the coefficients ([4], [5]). Several authors have investigated the problem of bounding g from below. Il'in, Kalashnikov, and Oleinik [5] proved that $g \geqq \text{const } (t-\tau)^{-n/2}$ in the paraboloid $|x-\xi|^2 \leqq \text{const}(t-\tau)$; while Besala [3] and Friedman [4] have derived lower bounds for g which are valid when $t-\tau$ is bounded away from zero. In the appendix to his important paper [6] on Hölder continuity of solutions of parabolic and elliptic equations, Nash asserts the existence of global upper and lower bounds for the fundamental solution of the divergence structure parabolic equation

$$(2) \qquad u_t - \{a_{ij}(x, t)u_{x_i}\}_{x_j} = 0.$$

These bounds do not involve the fundamental solutions of equations of the form $\alpha\Delta u = u_t$ and thus are not as sharp as the classical upper bound for (1). On the other hand, Nash's estimates are independent of the moduli of continuity of the coefficients of (2).

For (x, t) in the strip $S = E^n \times (0, T)$, consider the parabolic equation

$$(3) \qquad u_t - \{a_{ij}(x, t)u_{x_i} + a_j(x, t)u\}_{x_j} - b_j(x, t)u_{x_j} - c(x, t)u = 0.$$

Assume that the coefficients of (3) are bounded measurable functions of (x, t) in S and that there exists a constant $\nu > 0$ such that $a_{ij}(x, t)\zeta_i\zeta_j \geqq \nu^{-1}|\zeta|^2$ almost everywhere in S for all $\zeta \in E^n$. Let $g(x, t; \xi, \tau)$ denote

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