SOME REMARKS ON PARALLELIZABLE STEIN MANIFOLDS

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1. The purpose of this note is to collect some simple facts on the parallelizability of analytic submanifolds of the complex number space C^N, which are remarkable because their analogues in the real case fail to be true. Any analytic submanifold of C^N is a Stein manifold. (An analytic submanifold is closed in C^N by definition.) Conversely, every Stein manifold can be embedded in some C^N, i.e. mapped biholomorphically onto an analytic submanifold of C^N. An n-dimensional Stein manifold X is called parallelizable if there exists a holomorphic field of n-frames on X, i.e. n holomorphic vector fields which are linearly independent at every point $x \in X$. (We require throughout this paper that all connected components of a manifold have the same dimension.) By a theorem of Grauert [2], an n-dimensional Stein manifold X is parallelizable if and only if there exists a continuous field of (complex) n-frames on X. We connect the parallelizability with the notion of complete intersection: An n-dimensional analytic submanifold X of C^N is called a complete intersection, if the ideal $I(X)$ of all holomorphic functions on C^N which vanish on X can be generated by $N-n$ elements. This is the case if and only if there exist $N-n$ holomorphic functions $f_1, \cdots, f_{N-n}$ on C^N such that

$$X = \{x \in C^N : f_1(x) = \cdots = f_{N-n}(x) = 0\}$$

and the rank of the functional matrix of $(f_1, \cdots, f_{N-n})$ equals $N-n$ at every point $x \in X$. We shall prove that a Stein manifold is parallelizable if and only if it can be embedded as a complete intersection in some complex number space C^N.

2. The following lemma expresses the duality between the normal and tangent bundle of an analytic submanifold of C^N.

LEMMA. *Let X be an n-dimensional analytic submanifold of C^N.*
(i) *If the normal bundle of X is trivial, then X is parallelizable.*
(ii) *If X is parallelizable and $N \geqq 3n/2$, then the normal bundle of X is trivial.*

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