COMPLEX STIEFEL MANIFOLDS, SOME HOMOTOPY GROUPS AND VECTOR FIELDS[1]

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Introduction. Throughout this note M will denote a $\mathcal{C}^\infty$, closed, compact, connected, $2n$-dimensional, almost-complex manifold. That is, M is an orientable manifold and there is a reduction of the structure group of the tangent bundle of M from $SO(2n)$ to $U(n)$.

Let real (complex) span (M) denote the maximal number of vector fields on M which are linearly independent over the real (complex) numbers. We obtain certain lower bounds on real (complex) span (M).

Our method is that of obstruction theory in the universal example for such a problem, namely a fibration of the form $W_{n+k,k}\to BU(n) \to BU(n+k)$. Here $W_{n+k,k}$ is the Stiefel manifold of complex k-frames in complex $(n+k)$-space, and $BU(l)$ is the classifying space for complex l-plane bundles. We use the results of Mahowald [2] and Thomas [5] in the obstruction-theoretic problems encountered. The work is done in the category of vector bundles over *cw*-complexes and is then specialized to the case of manifolds.

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Homotopy groups. We give tables of homotopy groups of $W_{n+k,k}$ useful in doing the obstruction theory. More homotopy groups are given than are actually needed.

It is a classically known fact that $W_{n+k,k}$ is $2n$-connected. Combining this with the fact that, for $p\leqq 2(k-1)$ a positive integer, $\pi_{2n+p+1}(W_{n+k,k})\approx\pi_{2n+p+1}(W_{n+k+l,k+l})$ for $l\geqq 1$ an integer, we define $\pi_{2n+p+1}(W_{n+k,k})$ to be the p-stem of the complex Stiefel manifolds, for p as above. We obtain essentially the first seven of these stems and some results on the 8 and 9-stems. Some unstable results appear in the tables for the sake of completeness. We remark that $W_{n+1,1}=S^{2n+1}$ and $W_{n,n}=U(n)$ and are not included in the tables, since the homotopy groups in our range of dimensions are well-known for these spaces. The tables list $\pi_{2n+p+1}(W_{n+k,k})$:

[1] These results are part of the author's doctoral thesis submitted to the University of California, Berkeley.