

## VARIETIES OF GROUPS

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**1. Introduction.** This lecture should really be given by Hanna Neumann, because her book with the same title is going to be published very shortly by Springer-Verlag, in the series "Ergebnisse der Mathematik"; and she gave an expository address with the same title at the Group Theory Conference at Canberra in August, 1965, and this also will be published very soon by Gordon and Breach in the Proceedings of that conference. All I can do is to follow the advice of that well-known junior mathematician Tom Lehrer and plagiarize.

**2. Varieties.** "Varieties" are equationally defined classes of algebras: starting from a "species" of algebras, that is the class of algebras with given numbers of algebraic operations of given kinds, unary, binary, and so on, we single out subclasses by imposing certain "laws" or "identical relations". Thus, for example, a single binary operation defines the species of "groupoids"; if the associative law is imposed on the operation, we obtain the variety of "semigroups". The study of varieties was initiated by Garrett Birkhoff in 1935; the name "variety" was introduced by Philip Hall in 1949. Much of the theory belongs to Universal Algebra, but I shall be mainly concerned with varieties of groups, because that is where the deepest theorems have been proved and the most difficult and intriguing problems remain open. To mention some examples of varieties of groups and to fix some notation:

$\mathfrak{G}$  is the variety of all groups;

$\mathfrak{E}$  is the variety of trivial (= one-element) groups;

$\mathfrak{A}$  is the variety of abelian groups, singled out by the commutative law  $xy = yx$ ;

$\mathfrak{B}_e$  is the "Burnside" variety of groups of exponent  $e$ , singled out by the law  $x^e = 1$ ;

$\mathfrak{A}_e = \mathfrak{B}_e \cap \mathfrak{A}$  is the variety of abelian groups of exponent  $e$ .

This last example incidentally illustrates the fact that an intersection of varieties is again a variety; not only an intersection of two

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<sup>1</sup> This is an amplified version of an address delivered on January 27, 1967, at the Annual Meeting, in Houston, by invitation of the Committee to Select Hour Speakers for Annual and Summer Meetings; received by the editors March, 1967.