EXTREMAL PROBLEMS FOR A CLASS OF FUNCTIONALS DEFINED ON CONVEX SETS[1]

BY ZEEV NEHARI

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1. Let $X=(X, \Sigma)$ be a measurable space, and let τ be a class of positive measures μ defined on Σ. We consider a set H of nonnegative functions belonging to $L^p(\mu)$ on X for all $\mu \in \tau (1 \leqq p < \infty)$, and we denote by $C(H)$ the convex hull of H. If σ is an arbitrary positive measure on X, we define the functional $\Lambda(r)$ $(r \in C(H), L^1(\sigma))$ by

$$\Lambda(r) = \sup_{\mu \in \tau} \left[\int_X r^p d\mu \right]^{1/p} \Big/ \int_X r d\sigma. \tag{1}$$

The following result is a useful tool in the treatment of numerous extremal problems involving eigenvalues of differential and integral equations.

THEOREM I. *If* $\Lambda(r)$ *is the functional defined by* (1), *then*

$$\sup_{r \in C(H)} \Lambda(r) = \sup_{s \in H} \Lambda(s). \tag{2}$$

The proof of (2) is very simple. Since $H \subseteq C(H)$, (2) will follow from the inequality

$$\sup_{r \in C(H)} \Lambda(r) \leqq \sup_{s \in H} \Lambda(s), \tag{3}$$

and it is sufficient to establish (3) for finite sums of the form

$$r = \alpha_1 s_1 + \cdots + \alpha_n s_n, \quad \alpha_k > 0, \quad \sum_{k=1}^n \alpha_k = 1, \quad s_k \in H. \tag{4}$$

By Minkowski's inequality, we have

$$\left[\int_X r^p d\mu \right]^{1/p} \leqq \sum_{k=1}^n \alpha_k \left[\int_X s_k^p d\mu \right]^{1/p}$$

and thus, by (1),

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