THE RANGE OF A MEASURE

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Let X be a measure space, and λ a finite measure (countably additive) with values in R^n. The range of λ is the set of all vectors $\lambda(E)$ where E ranges over the measurable subsets of X. In a previous paper [2] we introduced a certain positive measure ν, determined by the measure λ, on the projective space P^{n-1}. We then showed that a necessary and sufficient condition that the range of λ be a ball is that ν be an orthogonally invariant measure on P^{n-1}. Our purpose in this paper is to show that that result was but a special case of a more general theorem: If X' is another measure space, with λ' a finite measure on X' with values in R^n, and if ν' is the corresponding positive measure on P^{n-1}, then a necessary and sufficient condition that the range of λ and the range of λ' have the same convex hull is that (a) $\lambda(X) = \lambda'(X')$ and (b) $\nu = \nu'$. If only condition (b) is satisfied, the range of λ is a translate of the range of λ'.

The appearance of the convex hull of the range of the measure is not surprising, because, as shown by Liapunoff, if the measure is atom free its range is a convex set (see [2]). The theorem is false if we refer only to the range of the measures, and not to the convex hull of the range, as can easily be shown by example.

For x a point in S^{n-1}, we denote by H_x the hemisphere $\{y \in S^{n-1}; \langle y, x\rangle \geqq 0\}$.

LEMMA 1. *Let μ be a real Borel measure on S^{n-1}. Let α be the natural map of S^{n-1} onto P^{n-1}, and let ν be the Borel measure on P^{n-1} defined by $\nu(E) = \mu(\alpha^{-1}(E))$. Then a necessary and sufficient condition that for each x in S^{n-1}*

$$\int_{H_x} \langle x, y\rangle \mu(dy) = 0$$

is that (a) $\int_{S^{n-1}} y\mu(dy) = 0$ *and* (b) ν *is the zero measure on* P^{n-1}.

PROOF. First assume that (a) and (b) are satisfied. Condition (a) implies that

$$\int_{H_x} \langle x, y\rangle \mu(dy) + \int_{H_{-x}} \langle x, y\rangle \mu(dy) = 0$$

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