A NOTE ON MINIMAL VARIETIES[1]

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1. Introduction. In [1] Almgren considered the situation of a closed minimal variety H, of dimension 2 immersed in S^3. He observed that the second fundamental form, a real valued bilinear form on the tangent space to H, is in fact the real part of a holomorphic quadratic differential with respect to the conformal structure on H induced by the metric inherited from its immersion in S^3. He used this fact to conclude that S^2 could not be immersed as a minimal variety in S^3 unless it was already totally geodesic.

It turns out that under the most general circumstances the second fundamental form of a p-dim minimal subvariety of an n-dim Riemannian manifold satisfies a natural second-order elliptic differential equation which generalizes the holomorphic condition mentioned above. In the case that the ambient manifold is S^n the equation may be used to show that a closed minimal subvariety of S^n, of arbitrary codimension, which does not twist too much is already totally geodesic. In a sense this theorem is analogous to Bernstein's theorem for complete minimal subvarieties in R^n.

2. A standard operator. Let M be a Riemannian manifold[2] of dimension n and $V(M)$ a d-dimensional vector bundle over M. Suppose the fibers of $V(M)$ carry a euclidean inner product and suppose there is given a connection in $V(M)$ *which preserves this inner product.* If W is a cross-section in $V(M)$ and $x \in T(M)_m$, the tangent space to M at m, we denote by $\nabla_x W$ the covariant derivative of W in the x direction. $\nabla_x W \in V(M)_m$.

Let $x, y \in T(M)_m$. We define $\nabla_{x,y} W \in V(M)$ as follows. Let Y be a vector field on M which extends y. We then set

$$\nabla_{x,y} W = \nabla_x \nabla_Y W - \nabla_{\nabla_x Y} W \tag{2.1}$$

where $\nabla_x Y$ is ordinary covariant differentiation of a vector field on M with respect to the Riemannian connection. It is easy to see that this definition is independent of the choice of Y.

Let $e_1, \cdots, e_n$ be an orthonormal basis of $T(M)_m$. If W is a cross-section in $V(M)$ we define $\nabla^2 W$ by

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[2] All manifolds will be assumed to be orientable.