QUASI-PERIODIC SOLUTIONS OF NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS WITH SMALL DAMPING

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A function $f(t)$ is called quasi-periodic if it can be represented in the form

$$f(t) = F(\omega_1 t, \omega_2 t, \cdots, \omega_m t)$$

where $F(\theta_1, \cdots, \theta_m)$ is a continuous function of period 2π in θ_ν, $\nu=1, \cdots, m$. The numbers $\omega_1, \cdots, \omega_m$ are called the basic frequencies of $f(t)$. We shall denote by $A(\omega_1, \cdots, \omega_m)$ the class of all functions f for which F is real analytic. For simplicity of notation we set $\theta=(\theta_1, \cdots, \theta_m)$ and $\omega=(\omega_1, \cdots, \omega_m)$ (then $A(\omega_1, \cdots, \omega_m)=A(\omega)$ and $F(\theta_1, \cdots, \theta_m)=F(\theta)$).

The purpose of this note is to study the family of complex systems of differential equations:

$$\begin{aligned} \dot{z} &= \lambda z + \epsilon f(t, z, \bar{z}), \\ \dot{\theta} &= \omega \end{aligned} \tag{1}$$

parametrized by λ, f analytic in z, $\bar{z}$, and $f \in A(\omega)$—i.e. $f(t, z, \bar{z}) = g(\theta, z, \bar{z})$ where g is 2π-periodic in θ—to determine the complex numbers, λ, for which there exists a solution $z=\phi(t, \epsilon) \in A(\omega)$.[1]

For $\operatorname{Re} \lambda=0$ there may be no solutions even in the linear case

$$\begin{aligned} \dot{z} &= \lambda z + \epsilon g(\theta), \\ \dot{\theta} &= \omega \end{aligned} \tag{2}$$

because of resonance. It is well known that if $\operatorname{Re} \lambda \neq 0$ and $\epsilon>0$ is small compared with $|\operatorname{Re} \lambda|$ then (1) always has a solution $z=\phi(t, \epsilon) \in A(\omega)$. This was shown by Malkin [7] and Bohr and Neugebauer [4] in the linear case and by Stoker [10] and, in the general case, by Bogoliubov [1].

Our main interest is $|\operatorname{Re} \lambda|$ small compared to ϵ. We shall describe a domain, Ω, in the λ-plane such that for each $\lambda \in \Omega$ the corresponding system (1) has a solution $z=\phi(t, \epsilon) \in A(\omega)$. We call Ω a nonresonance domain. We will show that Ω contains in particular $|\operatorname{Re} \lambda| > 1$ (this

[1] This system is derived from the second order equation $\ddot{x}+c\dot{x}+ax=f(t, x, \dot{x})$ (f quasi-periodic in t) by the transformation $z=\dot{x}+\alpha x$ for some constant α.