# THE VIRIAL THEOREM AND ITS APPLICATION TO THE SPECTRAL THEORY OF SCHRÖDINGER OPERATORS

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**1. Introduction.** Let us consider elliptic differential operators of the form

$$H = -\Delta + q(x), \qquad x \in R^m,$$

where the potential $q(x)$ satisfies the following conditions:

(I) $q \in Q_\alpha(R^m)$ for some $\alpha > 0$; i.e.

$$M_q(x) = \int_{|x-y| \leq 1} |q(y)| \, |x-y|^{4-m-\alpha} dy$$

is uniformly bounded for $x \in R^m$.

(II) For every $x \in R^m$, $x \neq 0$, there exists a radial derivative $q_r(x)$ of $q(x)$ and

$$\epsilon^{-1} |q((1+\epsilon)x) - q(x)| \leqq q_0(x) \in Q_\beta(R^m)$$

holds for $0 < \epsilon < \epsilon_0$ and some $\beta > 0$; in particular we have $rq_r(x) \leqq q_0(x)$; hence $rq_r \in Q_\beta(R^m)$.

Under these conditions we shall prove in §2 a very general form of the Virial Theorem of quantum mechanics. In §§3 and 4 this theorem will be used to deduce some results on the spectrum of $H$.

Let $L_2(R^m)$ be the Hilbert space of functions which are square-summable over $R^m$; the inner product in this space will be denoted by $\langle \cdot , \cdot \rangle$, the norm by $|\cdot|$.

From condition (I) one can conclude (e.g. Ikebe-Kato [2]):

(1) The operator $H$ with domain $D(H) = H_2(R^m)$ is selfadjoint in $L_2(R^m)$ ($H_2(R^m)$ is the closure of $C_0^\infty(R^m)$ with respect to the norm $|u|_2 = \{ \sum_{j,k} |\partial^2 u/(\partial x_j \partial x_k)|^2 + \sum_j |\partial u/\partial x_j|^2 + |u|^2 \}^{1/2}$).

(2) For $u \in D(H)$ and $q \in Q_\alpha(R^m)$ we have $qu \in L_2(R^m)$.

(3) For $u, v \in D(H)$ we have $\Delta u, \Delta v \in L_2(R^m)$ and $\langle \Delta u, v \rangle = \langle u, \Delta v \rangle$.

## 2. The Virial Theorem.

THEOREM. *Let conditions* (I) *and* (II) *be satisfied. If* $\lambda$ *is an eigenvalue of* $H$, $u(x)$ *a corresponding eigenfunction, then*

$$\langle (2q + rq_r - 2\lambda)u, u \rangle = 0, \qquad 2\langle -\Delta u, u \rangle = \langle rq_r u, u \rangle.$$