A GEOMETRIC PROOF OF RYLL-NARDZEWSKI'S FIXED POINT THEOREM

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In [4], Ryll-Nardzewski gave what he called an 'old-fashioned' proof of his famous fixed point theorem. The purpose of the present note is to give an even more old-fashioned proof of the fixed point theorem. In fact, our proof uses nothing more than a category argument and the classical Krein-Milman theorem. Our terminology and notation shall be those of Kelley, Namioka et al. [2]. The following geometric lemma is essential to our proof of Ryll-Nardzewski's fixed point theorem. In case the space E and the pseudo-norm p in the lemma are a Banach space and its norm respectively, the lemma is an easy consequence of Lindenstrauss' work [3].[1]

LEMMA. *Let* $(E, \mathfrak{I})$ *be a locally convex Hausdorff linear topological space, let* K *be a nonempty* $\mathfrak{I}$*-separable, weakly compact, convex subset of* E*, and let* p *be a continuous pseudo-norm on* E*. Then for each* $\epsilon > 0$*, there is a closed convex subset* C *of* K *such that* $C \neq K$ *and* $p\text{-diam}(K \sim C) \leqq \epsilon$*, where, for any subset* X *of* E*,* $p\text{-diam}(X) = \sup\{p(x-y) : x, y \in X\}$.

PROOF. Let $S = \{x : p(x) \leqq \epsilon/4\}$; then S is a weakly closed convex body. Let D be the weak closure of the set of all extreme points of K. Since K is $\mathfrak{I}$-separable, a countable number of translates of S cover K and hence D. Since D is weakly compact, it is of the second category in itself with respect to the relative weak topology. Therefore there are a point k of K and a weakly open subset W of E such that $(S+k) \cap D \supset W \cap D \neq \emptyset$. Let K_1 be the closed convex hull of $D \sim W$, and let K_2 be the closed convex hull of $D \cap W$. Then, by the Krein-Milman theorem and the compactness of K_1 and K_2, K is the convex hull of $K_1 \cup K_2$. Furthermore $K_1 \neq K$. For, otherwise, by Theorem 15.2 of [2], $D \sim W$ would contain all the extreme points of K, contradicting the fact that $W \cap D \neq \emptyset$. Obviously $p\text{-diam}(K_2) \leqq \epsilon/2$. Now let r be a real number in $(0, 1]$ and let f_r be the map $K_1 \times K_2 \times [r, 1] \to K$ defined by $f_r(x_1, x_2, \lambda) = \lambda x_1 + (1-\lambda)x_2$. Then clearly the image C_r of f_r is weakly closed, and it is easy to check that

[1] After the draft of the present note was completed we learned that Professor J. L. Kelley knew independently that a lemma of this sort was needed for a proof of Ryll-Nardzewski's fixed point theorem. Thus he was able to give a short proof of the fixed point theorem for Banach spaces using Lindenstrauss' result.