ON THE RADIAL PROJECTION IN NORMED SPACES

BY D. G. DeFIGUEIREDO AND L. A. KARLOVITZ[1]

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1. Let X be a real normed space with norm $\| \ \|$, T the *radial projection mapping* defined by

$$Tx = x, \qquad \text{if } \|x\| \leqq 1, \quad \text{and} \quad Tx = x/\|x\|, \qquad \text{if } \|x\| \geqq 1.$$

Our concern is with the Lipschitz constant of T; i.e. with the constant K such that $\|Tx-Ty\| \leqq K\|x-y\|$ for all x, y in X. In particular, we wish to determine under what conditions on the space X the mapping T will be *nonexpansive*, i.e. $K=1$.

T is a special case of a proximity mapping defined by a convex set in a normed vector space, i.e. a mapping which assigns to each point of X, the nearest point of the convex set C. There has been a good deal of interest in recent years in proximity maps, nonexpansive mappings, and their interrelations (Moreau, Browder, Petryshyn, Kirk, De Prima, Lions and Stampacchia, and others). It is easy to see that if X is an inner product space (and in particular, a Hilbert space), then T and every proximity map is nonexpansive. More precisely, Kirk and Smiley [1] proved that X is an inner product space if and only if for all nonzero x and y in X

$$\|x/\|x\| - y/\|y\|\,\| \leqq 2/(\|x\| + \|y\|)\|x - y\|.$$

For an arbitrary normed space X, Dunkl and Williams [2] have proved that for all nonzero x, y in X

$$\|x/\|x\| - y/\|y\|\,\| \leqq 4/(\|x\| + \|y\|)\|x - y\|.$$

From this it can be seen that $K \leqq 2$. This bound is the best possible because it is easily seen that for l_1, $K=2$.

It is of great interest in nonlinear functional analysis to know if there is a normed space which is not an inner product space and for which $K=1$. We prove the following theorem which shows that such spaces exist only for the trivial case of dimension two.

THEOREM. *If X has dimension not less than three, then X is an inner product space if and only if T is nonexpansive.*

If X has dimension two then the nonexpansiveness of T does not imply that X is an inner product space.

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