INTRINSIC METRICS ON COMPLEX MANIFOLDS

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1. Definition of intrinsic pseudometric. Let M be a (connected) complex manifold. We shall define a pseudometric d on M in a natural manner so that it depends only on the complex structure of M and nothing else.

Let D be the open unit disk in the complex plane and ρ the distance on D defined by the Poincaré-Bergman metric of D. Given two points p and q of M, choose the following objects:

(1) points $p=p_0, p_1, \cdots, p_{k-1}, p_k=q$ of M and

(2) points $a_1, \cdots, a_k, b_1, \cdots, b_k$ of D and holomorphic mappings $f_1, \cdots, f_k$ of D into M such that $f_i(a_i)=p_{i-1}$ and $f_i(b_i)=p_i$ for $i=1, \cdots, k$.

For each choice of points and mappings satisfying (1) and (2), consider the number $\rho(a_1, b_1)+ \cdots +\rho(a_k, b_k)$. Let $d(p, q)$ be the infimum of the numbers obtained in this manner for all possible choices. It is easy to verify that d is a pseudometric on M in the sense that

$$d(p, q) \geqq 0, \quad d(p, q) = d(q, p), \quad d(p, q) + d(q, r) \geqq d(p, r)$$

for $p, q, r \in M$. The following two propositions are immediate from the definition of d.

PROPOSITION 1. *Let M and N be two complex manifolds and d_M and d_N the intrinsic pseudometrics of M and N. Then every holomorphic mapping $f: M \rightarrow N$ is distance-decreasing in the sense that*

$$d_M(p, q) \geqq d_N(f(p), f(q)) \quad \text{for } p, q \in M.$$

In particular, every holomorphic transformation of M is distance-preserving with respect to d_M.

PROPOSITION 2. *For the complex Euclidean space C^n, the pseudometric d is trivial, i.e., $d(p, q)=0$ for all $p, q \in C^n$.*

The following proposition follows from the Schwarz-Pick lemma.

PROPOSITION 3. *For the unit disk D, the pseudometric d coincides with the distance ρ defined by the Poincaré-Bergman metric.*

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