THE TROTTER PRODUCT FORMULA FOR PERTURBATIONS OF SEMIBOUNDED OPERATORS

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1. Introduction. Let A and B be linear operators generating semigroups $\exp(tA)$ and $\exp(tB)$ in a Hilbert space. Then under suitable conditions,

$$\exp(t(A + B)) = \lim_{n\to\infty} (\exp((t/n)A) \exp((t/n)B))^n.$$

This is the Trotter [6] product formula. Though it is a theorem in perturbation theory, it is related to the Feynman path integral representation of solutions of partial differential equations and provides the best mathematical realization of this idea presently known.

Feynman [1] considered the Schrödinger equation of quantum mechanics:

$$i(du(t)/dt) = -(1/2m)\Delta u(t) + Vu(t)$$

for $u(t)$ in $L^2(R^3)$, for each t, and with initial condition $u(0) = u$. Here Δ is the Laplace operator, and V is a real valued function on R^3. The solution when $V=0$ is

$$\begin{aligned} u(x, t) &= (\exp((it/2m)\Delta)u)(x) \\ &= (2\pi it/m)^{-3/2} \int \exp[i\tfrac{1}{2}m(|x - y|^2/t)]u(y)dy. \end{aligned}$$

If we take $A = (i/2m)\Delta$ and $B = -iV$, then as Nelson [4] observed, the Trotter formula gives

$$\begin{aligned} u(x, t) &= (\exp((it/2m)\Delta - itV)u)(x) \\ &= \lim_{n\to\infty} (\exp((it/2mn)\Delta) \exp(-(it/n)V))^n u(x) \\ &= \lim_{n\to\infty} \int \cdots \int \exp\left[i \sum_{j=1}^{n} \left\{ \frac{1}{2} m \frac{|x_j - x_{j-1}|^2}{(t/n)^2} \right.\right. \\ &\qquad \left.\left. - V(x_{j-1}) \right\} \frac{t}{n} \right] u(x_0)(2\pi it/nm)^{-3n/2} dx_0 \cdots dx_{n-1}, \end{aligned}$$

where $x_n = x$, as a representation of the solution of the full Schrödinger equation.