CONJECTURES CONCERNING ELLIPTIC CURVES

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1. The elliptic curve

$$\Gamma_D: X^3 + Y^3 = DZ^3,$$

where D in a cube-free positive integer, admits complex multiplication by $(-3)^{1/2}$. By the Mordell-Weil theorem, the group of rational points on Γ_D has a finite number of independent generators of infinite order, g say. The zeta-function of Γ_D has the form

$$\zeta(s)\zeta(s-1)/L_D(s),$$

where $\zeta(s)$ is the Riemann zeta-function and $L_D(s)$ is a Hecke L-series. Denote by $L'_D(s)$ the derivative of $L_D(s)$ with respect to s.

This note is a description of some numerical results obtained for the values of $L_D(1)$ and $L'_D(1)$ for many D, with special reference to the conjectures of Birch and Swinnerton-Dyer, [1]. In particular, when $g=1$, the value of $L'_D(1)$ is compared with a canonical measure for the density of the rational points on Γ_D. With the aid of further computations of second and third derivatives of $L_D(s)$ for a few values of D, a relation can be conjecturally formulated as

$$L_D^{(g)}(1)/f = g!\gamma\kappa/\eta^2.$$

Here, f is a product of factors due to "bad" primes, γ the order of the Tate-Šafarerič group, κ the inverse of the measure of the density, and η the number of points on Γ_D of finite order.

2. The conjectures of Birch and Swinnerton-Dyer, [1], are stated for general elliptic curves, especially for those which admit complex multiplication. They will be restated here for the curve Γ_D only.

CONJECTURE 1. *$L_D(s)$ has a zero at $s=1$ of order precisely g.*

For all cube-free D of the form 2^r3^sM, where $r, s=0, 1, 2$ and where M is such that the product of its distinct prime divisors ($\neq 2, 3$) is less than 100 (676 D in all), the values of $L_D(1)$ and $L'_D(1)$ were computed from approximation formulae. It can be shown that $L_D(1)$ is the product of a rational integer and a predictable factor, so that any

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