ON MANIFOLDS WITH INVOLUTION

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We consider a smooth involution ω on a smooth closed n-manifold M from the bordism point of view, as in [2, Chapter IV]. We know that the fixed-point set of ω is the disjoint union of submanifolds; let k be the maximum dimension of these. It is clear that if ω is free, M bounds a 1-disk bundle over the orbit space. Now fix k, and let n, M, and ω vary. Conner and Floyd prove [2, Theorem (27.1)] that if M does not bound, n cannot be arbitrarily large. Their proof is nonconstructive, and fails to give an upper bound for n. We obtain the precise bound.

THEOREM 1. *Suppose k is the maximum dimension of the fixed-point submanifolds of the smooth involution ω on the closed nonbounding n-manifold M. Then $n \leqq 5k/2$ (if k is even) or $n \leqq (5k-1)/2$ (if k is odd). Further, if we are given that the unoriented cobordism class of M is indecomposable, then $n \leqq 2k+1$.*

Examples in the extremal dimensions are easily constructed. Take homogeneous coordinates $(x_0, x_1, x_2, \cdots, x_i, x_1', x_2', \cdots, x_i')$ on real projective $2i$-space P_{2i} $(i>0)$, and define the involution ω_i by

$$\omega_i(x_0, x_1, \cdots, x_i, x_1', \cdots, x_i') = (x_0, x_1, \cdots, x_i, -x_1', \cdots, -x_i').$$

Then the product involution $\omega_i \times \omega_j$ on $P_{2i} \times P_{2j}$ maps the hypersurface $H_{2i,2j}$ defined by the equation

$$x_0y_0 + x_1y_1 + \cdots + x_iy_i + x_1'y_1' + \cdots + x_i'y_i' = 0$$

into itself, where for clarity we take coordinates $(y_0, y_1, \cdots, y_j, y_1', \cdots, y_j')$ on P_{2j}, and assume $i \leqq j$. The fixed-point dimension of $\omega_i \times \omega_j | H_{2i,2j}$ is found to be $i+j-1$. The manifold $H_{2i,2j}$ has dimension $2i+2j-1$ and its cobordism class $[H_{2i,2j}]_2$ is indecomposable if and only if the binomial coefficient

$$\binom{i+j}{i}$$

is odd. We can always choose i and j satisfying this condition and $i+j=m$ whenever m is not a power of 2. As an example for the first assertion of Theorem 1, we take the product of many copies of the 5-dimensional example $(H_{2,4}, \omega_1 \times \omega_2 | H_{2,4})$, with possibly one copy of (P_2, ω_1).