A CLASS OF WEIGHT FUNCTIONS FOR WHICH TCHEBYCHEFF QUADRATURE IS POSSIBLE

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Communicated by R. C. Buck, June 13, 1966

1. **Introduction.** By a weight function $W(x)$ we mean a real valued nonnegative function on $[-1, 1]$ for which the proper or improper Riemann integral exists and has the value one. If the equations

$$\frac{1}{n}\sum_{i=1}^{n} x_{i,n}^{k} = \int_{-1}^{1} x^{k} W(x)\,dx, \qquad k = 1, \cdots, n, \tag{1}$$

have real solutions for all positive integers n, we say Tchebycheff quadrature is possible for the weight function $W(x)$. A result attributed to Hermite [1] is that Tchebycheff quadrature is possible for

$$W(x) = \frac{1}{\pi(1-x^2)^{1/2}},$$

but the author has found no other weight function having this property in previous literature. The object of this note is to announce the explicit structure of an infinite family of Tchebycheff weight functions, and to sketch the proof. Full details will appear elsewhere.

2. THEOREM. *Tchebycheff quadrature is possible for*

$$W(x) = \frac{1}{\pi(1-x^2)^{1/2}}\,\frac{1+2ax}{1+4a^2+4ax}, \tag{2}$$

where $-1/4 \leqq a \leqq 1/4$.

SKETCH OF PROOF. Lemma 1 describes a general method for investigating the solutions of equations (1). Lemma 2 derives a special fact that facilitates the use of Lemma 1 for the case of the weight function defined by (2). Lemma 3 combines the previous information to complete the proof.

LEMMA 1. *Let $W(x)$ be a weight function and let $m_k = \int_{-1}^{1} x^k W(x)\,dx$, $k = 0, 1, \cdots$. Let*

$$f(z) = z\left(-\sum_{k=1}^{\infty} \frac{m_k}{kz^k}\right). \tag{3}$$

For a positive integer n let $T_n(z)$ be the polynomial part, including