FOURIER SERIES WITH POSITIVE COEFFICIENTS

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I shall state a number of results on sine or cosine series with nonnegative coefficients; proofs of these and some related theorems will appear elsewhere.

The following theorems are known.

A [1], [8]. *If* $\lambda_n \downarrow 0$, $\phi(x) = \sum \lambda_n \cos nx$, *and* $0 < \gamma < 1$, *then* $\sum n^{\gamma-1}\lambda_n < \infty$ *if and only if* $x^{-\gamma}\phi(x) \in L$.

A′ [6]. *If* λ_n *are the Fourier coefficients of* ϕ, $\lambda_n \geqq 0$, *and* $1 < \gamma < 3$, *then* $\sum n^{\gamma-1}\lambda_n < \infty$ *if and only if* $x^{-\gamma}[\phi(x) - \phi(0)] \in L$.

B [2], [3]. *If* $\lambda_n \downarrow 0$, $\phi(x) = \sum \lambda_n \cos nx$, $1 < p < \infty$, *and* $(1-p)/p < \gamma < 1/p$, *then* $x^{-\gamma}\phi(x) \in L^p$ *if and only if* $\sum n^{p+p\gamma-2}\lambda_n^p < \infty$.

C [7]. *If* $\lambda_n \downarrow 0$, $\phi(x) = \sum \lambda_n \cos nx$, *and* $0 < \gamma < 1$, *then* $\phi(x) \in \mathrm{Lip}\,\gamma$ *if and only if* $\lambda_n = O(n^{-\gamma-1})$.

There are similar theorems for sine series.

The following theorems generalize A and C (with different necessary and sufficient conditions), to series with nonnegative coefficients, and give a result that is related to B as A′ is related to A.

THEOREM 1. *If* $\lambda_n \geqq 0$, λ_n *are the Fourier sine or cosine coefficients of* ϕ *and* $0 < \gamma < 1$, *then*

$$\sum n^{\gamma-1}\lambda_n < \infty \tag{1}$$

if and only if

$$\int_{a+}^{\pi} (x-a)^{-\gamma}\phi(x)dx \text{ converges}, \quad 0 \leqq a < \pi. \tag{2}$$

More precisely, (1) is necessary for (2) with $a = 0$ and sufficient for (2) for all a—an illustration of the principle that a Fourier series with nonnegative coefficients tends to behave as well at all points as it does at 0. (The case $a = 0$ is a special case of a more general result of Edmonds [4, p. 235].) Theorem A′ can be generalized in the same way if $1 < \gamma < 2$.

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