ON THE MAXIMAL RING OF QUOTIENTS OF $C(X)$

BY ANTHONY W. HAGER

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1. Let $Q(X)$ denote the maximal ring of quotients (in the sense of Johnson [4] and Utumi [5]) of the ring $C(X)$ of continuous real-valued functions on the completely regular Hausdorff space X. This ring has been studied by Fine, Gillman, and Lambek [1] and realized by them as the direct limit of the subrings $C(V)$, V a dense open subset of X (i.e., the union of these $C(V)$'s, modulo the obvious equivalence relation). From this representation of $Q(X)$, it follows that if X and Y have homeomorphic dense open subsets, then $Q(X)$ and $Q(Y)$ are isomorphic. The full converse to this is false (see below). In this note a proof of the following is described.

THEOREM 1. *Let X and Y be separable metric spaces. If $Q(X)$ and $Q(Y)$ are isomorphic, then X and Y have homeomorphic dense open subsets.*

In particular, the spaces R^n, $n=1, 2, \cdots$ ($R=$the reals) have pairwise nonisomorphic Q's, thus settling a question[1] raised in [1]. That $Q(R)$ is not isomorphic to $Q(R^n)$, for $n>1$, was shown by F. Rothberger and J. Fortin. (See [2], and [1, p. 16].)

The main purpose of this note is to present a fairly simple solution to this question, and therefore the possible generalizations of Theorem 1 will not be discussed here. These generalizations, and related questions, will be treated in detail in a later paper.

The proof of Theorem 1 will now be described.

2. Homomorphisms of $C(Y)$ into $C(X)$ are well understood [3, Chapter 10]. If $\tau: X\rightarrow Y$ is continuous, $\phi(f)=f\circ\tau$ defines a homomorphism $\phi: C(Y)\rightarrow C(X)$. Conversely, if Y is realcompact, and $\phi: C(Y)\rightarrow C(X)$ is a homomorphism with $\phi(1)=1$, then ϕ is induced by a continuous function in this manner.

Now, let W_0 be a dense open subset of X, and let $\tau: W_0\rightarrow Y$ be continuous and additionally satisfy: for each dense open subset V of Y, $\tau^{-1}[V]$ is dense in X. Then $\phi(f)=f\circ\tau$ defines a homomorphism $\phi: Q(Y)\rightarrow Q(X)$. Evidently, ϕ satisfies

(*) for each dense open subset V of Y, there is a dense open subset W of X such that $\phi[C(V)]\subset C(W)$.

[1] The author is indebted to Professor Nathan J. Fine for communicating this question, and for many valuable conversations concerning it.