THE TREATMENT OF PERIODIC ORBITS BY THE METHODS OF FIXED POINT THEORY

BY F. BROCK FULLER

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Let M be a C^∞ manifold and let F be a C^∞ field of tangent velocity vectors defined over M. The question considered here is this: Is it possible to define an "index" for the periodic orbits of the differential equation $dx/dt = F(x)$ which has properties like those of the fixed point index [3]?

It is worthwhile to examine first the special case in which the velocity field F has a surface of section [1] because in this case the periodic orbits can be made to correspond to fixed points. Assume that M is compact and that it is possible to define a C^∞ mapping θ of M onto the circle of real numbers modulo 1 in such a way that $d\theta/dt > 0$ on every trajectory of F. Let S denote the surface of section $\theta = 0$ and let T denote the return map of S onto itself, obtained by following each point of S out along its positive trajectory until its first return to S. Define the *degree* of a periodic orbit to be the number of complete revolutions of θ as the orbit is traversed. Then to each periodic orbit of degree d corresponds a sequence x, $T(x), \cdots, T^{d-1}(x)$ of fixed points of T^d. These fixed points are distinct if the periodic orbit is simple, i.e. has least period. On the other hand if the periodic orbit has multiplicity m, by which we mean that it traverses a simple orbit m times, then exactly d/m of the fixed points are distinct. To the set of all periodic orbits of degree d we can assign the Lefschetz number $\Lambda(T^d)$, or total index of the fixed points of T^d. Now suppose that x is an isolated fixed point of T^d. The local behavior of T^d at the other fixed points $T(x), \cdots, T^{d-1}(x)$ is isomorphic to that at x, in particular all these fixed points are isolated and with the same index $i(x)$. If the corresponding orbit of degree d is simple, its contribution to $\Lambda(T^d)$ is then $d \cdot i(x)$; while if it is of multiplicity m, only d/m of the fixed points are distinct, so that its contribution to $\Lambda(T^d)$ is reduced by the factor $1/m$.

The total index $\Lambda(T^d)$ assigned to the periodic orbits of degree d can be refined by examining the free homotopy classes in M of the periodic orbits. The following relation can be shown: Two fixed points of T^d belong to the same Nielsen class [7] if and only if the periodic orbits of degree d on which they lie belong to the same free homotopy class. Note that the degree is an invariant of the free homotopy class. Thus to each free homotopy class of periodic orbits may be assigned the total index of a Nielsen class.