NORM INEQUALITIES FOR SOME ORTHOGONAL SERIES

BY RICHARD ASKEY[1]

To Professor A. Zygmund

1. Introduction. Over forty years ago M. Riesz [31] proved a theorem that is still inspiring new work. We refer to his celebrated conjugate function theorem.

Let $f(e^{i\theta})$ be integrable on $(0, 2\pi)$ and form its Fourier series

$$f(e^{i\theta}) \sim \sum_{-\infty}^{\infty} c_n e^{in\theta}. \tag{1}$$

The conjugate function $\tilde{f}(e^{i\theta})$ to $f(e^{i\theta})$ is the function which in some sense has the expansion

$$f(e^{i\theta}) \sim -i \sum_{-\infty}^{\infty} (\operatorname{sgn} n) c_n e^{in\theta}. \tag{2}$$

M. Riesz showed that if $f \in L^p$, $1 < p < \infty$, then $\tilde{f} \in L^p$, so (2) is an ordinary Fourier series, and

$$\|\tilde{f}\|_p \leqq A_p \|f\|_p \tag{3}$$

where

$$\|f\|_p = \left[\int_0^{2\pi} |f(e^{i\theta})|^p d\theta\right]^{1/p}.$$

One reason for considering the conjugate function is that $f(e^{i\theta}) + i\tilde{f}(e^{i\theta})$ has an analytic extension to the interior of the unit circle. In addition to this, there were two other applications of this theorem that were made many years ago. The first actually dates back before this theorem to a result of Kolmogoroff. Kolmogoroff [23] showed that the partial sums of (1) could be obtained in terms of f and $\tilde{f}$. Using this M. Riesz proved the inequality

$$\|S_n\|_p \leqq A_p \|f\|_p \qquad (n = 0, 1, \cdots) \tag{4}$$

where $S_n(f) = \sum_{-n}^{n} c_k e^{ik\theta}$ and A_p is independent of f and n. From (4) it was easily shown that $\lim_{n\to\infty} \|S_n - f\|_p = 0$ for all $f \in L^p$, $1 < p < \infty$.

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