EXISTENCE THEORY FOR TWO POINT BOUNDARY VALUE PROBLEMS[1]

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We consider second order two point boundary value problems of the form:

$$y'' = f(t, y, y'), \quad a \leqq t \leqq b; \tag{1}$$

$$a_0 y(a) - a_1 y'(a) = \alpha, \quad |a_0| + |a_1| \neq 0; \tag{2}$$

$$b_0 y(b) + b_1 y'(b) = \beta, \quad |b_0| + |b_1| \neq 0. \tag{3}$$

Our basic result is the

THEOREM. *Let $f(t, u, u')$ have continuous derivatives which satisfy:*

$$\frac{\partial f(t, u(t), u'(t))}{\partial u} > 0, \quad \left| \frac{\partial f(t, u(t), u'(t))}{\partial u'} \right| \leqq M, \tag{4}$$

for some $M \geqq 0$, $a \leqq t \leqq b$ and all continuously differentiable functions $u(t)$. Let the constants a_i, b_i satisfy:

$$a_i \geqq 0, \quad b_i \geqq 0, \quad i = 0, 1; \quad a_0 + b_0 > 0. \tag{5}$$

Then a unique solution of (1), (2), (3) *exists for each* (α, β).

PROOF. We sketch the proof. The initial value problem

$$\begin{aligned} &u'' = f(t, u, u'), \quad a \leqq t \leqq b; \\ &\left.\begin{aligned} a_0 u(a) - a_1 u'(a) &= \alpha \\ c_0 u(a) - c_1 u'(a) &= s \end{aligned}\right\}, \quad a_1 c_0 - a_0 c_1 = 1; \end{aligned} \tag{6}$$

has the unique solution $u(s; t)$. The problem (1), (2), (3) has as many solutions as there are real roots, s^* of

$$\phi(s) \equiv b_0 u(s; b) + b_1 u'(s; b) = \beta.$$

Since $u(s; t)$ is continuously differentiable with respect to s the derivative $\xi(t) \equiv \partial u(s; t)/\partial s$ satisfies the variational problem [1],

$$\xi'' = p(t)\xi' + q(t)\xi,$$
$$\xi(a) = a_1, \quad \xi'(a) = a_0;$$

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