AHLFORS' CONJECTURE CONCERNING EXTREME SARIO OPERATORS

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The linear operators, introduced by Sario [4] to construct harmonic functions with prescribed properties on Riemann surfaces, form a convex set. Ahlfors [1] has conjectured a representation for the extreme operators of this convex set. We give an equivalent formulation of this conjecture and show that it is not true in general.

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1. Let W be a subregion of a Riemann surface R. We suppose W has a compact complement and that its relative boundary α is analytic. We consider a linear operator T which, to continuous values f on α, assigns a harmonic function Tf on W such that $Tf=f$ on α. T is assumed to have the following additional properties:

$$T1 = 1, \qquad Tf \geqq 0 \quad \text{if } f \geqq 0, \tag{1.1}$$

$$\int_\alpha \frac{\partial Tf}{\partial n}\, ds = 0. \tag{1.2}$$

Sario [4] has called these operators *normal linear operators*. It is clear that the set of such operators on W form a convex set.

2. We assume, with Ahlfors [1], that the ideal boundary β of R is analytic. Consider the harmonic measure of the region befween α and β. That is, the harmonic function on W which is 0 on α and 1 on β and normalized so that the period of its conjugate function along α is 1. In terms of this conjufate function we parametrize α and β by $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$, respectively.

Given f on α, Tf has radial limits almost everywhere on β and this null set E may be selected independent of f (See [1]). In this manner we may consider T as inducing a linear mapping from the space $C(0, 1)$ of continuous functions on α to $L^\infty(0, 1)$ the space of bounded measurable functions on β. We denote this induced linear operator by T also and the class of all such operators by L. They have the following properties corresponding to conditions (1.1) and (1.2) above:

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