ON POINCARÉ'S BOUNDS FOR HIGHER EIGENVALUES

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1. **Introduction.** Let A be a compact symmetric negative-definite operator on a real Hilbert space H having the inner product (u, v). Let $\lambda_1 \leqq \lambda_2 \leqq \cdots$ be the eigenvalues and $u_1, u_2, \cdots$ the corresponding orthonormal set of eigenvectors of the equation $Au = \lambda u$. Denote by $R(u)$ the Rayleigh quotient $(Au, u)/(u, u)$. For a given λ_n let m and N be the smallest and largest indices respectively such that $\lambda_m = \lambda_n = \lambda_N$. There are two variational characterizations of λ_n by inequalities. One goes back to Poincaré [**1**, p. 259] and was reformulated by Pólya and Schiffer [**2**], [**3**]. The other is the maximum-minimum principle for which A. Weinstein [**4**], [**5**] recently introduced a new approach. Using the Weinstein determinant and the corresponding quadratic form he gave for the first time a complete discussion of the corresponding inequalities including the necessary and sufficient conditions for equality. In the present paper we give a similar discussion of Poincaré's characterization of λ_n.

2. **The main result.** Let V_r be any r-dimensional subspace of H and let $p_1, p_2, \cdots, p_r$ be a basis for V_r. We consider the determinant

$$(1) \qquad \det\{(Ap_i, p_k) - \lambda(p_i, p_k)\}, \qquad i, k = 1, 2, \cdots, r.$$

Using Parseval's formula we see that (1) can also be written as

$$(2) \qquad \det\left\{\sum_{j=1}^{\infty}(\lambda_j - \lambda)(p_i, u_j)(p_k, u_j)\right\}, \qquad i, k = 1, 2, \cdots, r.$$

Let us note in passing the remarkable, but until now unexplained, similarity between (2) and the Weinstein determinant

$$(3) \qquad W(\lambda) = \det\left\{\sum_{j=1}^{\infty}(\lambda_j - \lambda)^{-1}(p_i, u_j)(p_k, u_j)\right\}, \quad i, k = 1, 2, \cdots, r.$$

We can now formulate our main result.

THEOREM. *For any choice of V_r we have the inequality*

[1] This paper was prepared by the author, while the author was an NDEA fellow in the Institute for Fluid Dynamics and Applied Mathematics of the University of Maryland.