ON EIGENVALUE DISTRIBUTIONS FOR ELLIPTIC OPERATORS WITHOUT SMOOTH COEFFICIENTS

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1. **Introduction.** An asymptotic formula for the eigenvalues of a self-adjoint realization of an elliptic operator on a bounded region has been obtained in various cases by a number of authors; see [1], [4] and the references there. In each case the method of proof demands more regularity of the coefficients of the operator (at least when the order of the operator is low relative to the dimension of the space) than should be necessary for validity of the formula. The purpose of this note is to derive the formula under minimum assumptions on the coefficients, in two cases: the Dirichlet realization in a compact manifold with boundary, and the unique realization in a compact manifold without boundary. We use the known results for operators with smooth coefficients and a simple abstract approximation method based on the "minimax" formula for eigenvalues.

2. **Abstract eigenvalue estimates.** Let H be a separable complex Hilbert space with inner product (u, v) and norm $\|u\|$. If S is a compact operator in H, we denote by $\{\mu_j(S)\}$ the sequence of eigenvalues of SS^*, with $\mu_1(S) \geqq \mu_2(S) \geqq \cdots$.

Suppose that S is a 1-1 compact normal operator in H. The range $R(S)$ can be considered as a Hilbert space K with inner product $\langle u, v\rangle = (S^{-1}u, S^{-1}v)$ and norm $|u|$. Let T be a second operator in H with $R(T) \subseteq K$. If J denotes the natural injection of K into H, then $S = JS_1$ and $T = JT_1$ with S_1 and T_1 operators from H to K. Since T_1 is closed, hence continuous, and J is compact, it follows that T is compact. We can estimate the eigenvalues $\mu_j(T)$ as follows:

LEMMA 1. *For all* $j = 1, 2, \cdots$,

$$\mu_j(T) \leqq (\|T_1\| + \|S_1\|)\|T_1 - S_1\|\mu_j(S) + \mu_j(S).$$

PROOF. Let $\{\phi_j\}$ be a complete orthonormal sequence in H consisting of eigenvectors of S, with corresponding eigenvalues $\{\lambda_j\}$ satisfying $|\lambda_1| \geqq |\lambda_2| \geqq \cdots$. Let H_j be the closed subspace of H spanned by $\{\phi_k;\ k \geqq j\}$. Applying the "minimax" formula [5, Theorem X.4.3] to the positive compact operator TT^* we have

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