ON MALLIAVIN'S COUNTEREXAMPLE TO SPECTRAL SYNTHESIS

BY IAN RICHARDS[1]

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Malliavin's disproof of spectral synthesis breaks into two main parts: the first uses a certain "operational calculus," while the second involves a construction. Here we will be concerned only with the second part. The required construction is complicated, and several versions of it have been given [2], [3], [5]. In this note we describe an approach which appears to be somewhat simpler.

It should be mentioned that Varopoulos [6], [7] has recently given a completely different disproof of spectral synthesis, using tensor products of Banach algebras. However Malliavin's original counterexample, although difficult to construct, gives a very powerful result: for instance it shows that spectral synthesis fails even for principal ideals.

(Malliavin's results imply the existence of an $f\in A(\Gamma)$ such that $f, f^2, f^3, \cdots$ all generate different closed ideals in $A(\Gamma)$. The hypothesis of spectral synthesis asserts that any two closed ideals in $A(\Gamma)$ having the same "zero set" coincide—see below for definitions.)

DEFINITIONS. G is an infinite discrete abelian group; Γ is its dual, which is compact and not discrete. For $f\in L^1(G)$ the Fourier transform $\hat{f}$ is defined by $\hat{f}(\gamma)=\sum_G f(p)(-p,\gamma)dp$. For $f\in L^1(\Gamma)$ we set $\hat{f}(p)=\int_\Gamma f(\gamma)(p,\gamma)d\gamma$, so that the Fourier inversion theorem holds: $f^{\wedge\wedge}=f$. $A(\Gamma)$ denotes the algebra of Fourier transforms $\hat{f}(\gamma), f\in L^1(G)$. It is endowed with the norm, $\|\hat{f}\|\equiv\sum|f(p)|$, so that $A(\Gamma)$ is just an isomorphic and isometric copy of $L^1(G)$. The *zero set* of an ideal $I\subseteq A(\Gamma)$ is the set of points $\gamma_0\in\Gamma$ such that $f(\gamma_0)=0$ for all $f\in I$. For $g\in A(\Gamma)$, $\eta(g)$ denotes the L^∞ norm of the sequence $\hat{g}$: $\eta(g)\equiv\sup|\hat{g}(p)|$, $p\in G$.

Malliavin showed (cf. [3], [4], [5]) that spectral synthesis fails in $L^1(G)$ provided there exists a real valued function $f\in A(\Gamma)$ such that, for $-\infty<u<\infty$

$$\text{(A)}\qquad \eta(e^{iuf}) = O(|u|^{-n}),\quad \text{all } n\geqq 0.$$

THEOREM. *There exists a real valued function $f\in A(\Gamma)$ which satisfies* (A).

PROOF. To simplify the discussion, we will assume that G is the group of integers (Γ is the circle group). The modifications needed for dealing with the general case are similar to those in Rudin [5,

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