DERIVATIONS OF LIE ALGEBRAS[1]

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1. It is known as a theorem of E. Schenkman and N. Jacobson that every nilpotent Lie algebra over a field of arbitrary characteristic has an outer derivation (see [1]). In connection with this theorem, we know the following two types of results, one showing a wider class of Lie algebras which have outer derivations and the other showing the existence of outer derivations in proper ideals of the derivation algebras. Namely, G. Leger [2] has shown that, if a Lie algebra over a field of characteristic 0 whose center is $\neq (0)$ has no outer derivations, it is not solvable and its radical is nilpotent. On the other hand, T. Satô [3] has shown that every nilpotent Lie algebra over a field of characteristic 0 has an outer derivation in the radical of its derivation algebra. We shall generalize and sharpen these results and give more detailed results on outer derivations of Lie algebras over a field of arbitrary characteristic.

2. We denote by $Z(H)$ the center of a Lie algebra H. Then we have

THEOREM 1. *Every Lie algebra L over a field Φ of arbitrary characteristic such that $L \neq L^2$ and $Z(L) \neq (0)$ has an outer derivation. More precisely, such a Lie algebra L has a nilpotent outer derivation D such that $D^2 = 0$, unless L is either 1-dimensional or the direct sum of a 1-dimensional ideal and of an ideal L_1 such that $L_1 = L_1^2$ and $Z(L_1) = (0)$.*

In the case where L is not abelian and has no abelian direct summands, take a subspace M of L of codimension 1 containing L^2. Then M is an ideal of L and $[L, Z(M)] \neq Z(M)$. Choose an element e of L such that $L = \Phi e + M$ and an element z of $Z(M)$ which is not in $[L, Z(M)]$. Then the endomorphism D of L defined in such a way that $De = z$ and $DM = (0)$ is an outer derivation of L such that $D^2 = 0$.

COROLLARY. *Let L be a Lie algebra over a field of characteristic 0 such that $Z(L) \neq (0)$ and R be the radical of L. If L has no outer derivations, L is not solvable and $R = [L, R]$.*

There is another class of nonsolvable Lie algebras which have outer derivations. Namely:

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