WIENER-HOPF TYPE PROBLEMS FOR ELLIPTIC SYSTEMS OF SINGULAR INTEGRAL EQUATIONS[1]

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The problem treated in this paper is roughly the inversion of elliptic systems of singular integral equations in a half-space of R^n. Ellipticity means that the system is invertible over the whole of R^n, in our case explicitly. We first introduce notation and some spaces of (vector-valued) distributions.

Let (x, y) denote points in R^n with $x \in R^{n-1}$, $y \in R$. R^n_+ $[R^n_-]$ is the half-space $y \geqq 0$ $[y \leqq 0]$. $H^{s,p}$ is the space of distributions u for which

$$\|u\|_{s,p} = \|F^{-1}(1 + |\xi|^2 + \eta^2)^{s/2} Fu\|_{L^p} < \infty.$$

Here $(Fu)(\xi, \eta) = \int u(x, y) e^{i(x\cdot\xi + y\cdot\eta)}\, dxdy$, with (ξ, η) dual to (x, y). We assume $1 < p < \infty$. $H^{s,p}_-$ is the subspace of elements supported in R^n_-. $H^{s,p}(R^n_+)$ is the quotient $H^{s,p}/H^{s,p}_-$ (it is a space of distributions on $\mathring{R}^n_+$, the open half-space). Y_+ denotes the canonical map onto the quotient and $\|Y_+ u\|_{s,p}$ is the quotient norm. $H^{s,p}_+$, $H^{s,p}(R^n_-)$ and Y_- are similarly defined. For $s = 0$, we can identify $H^{0,p}_\pm = L^p_\pm$ with $L^p(R^n_\pm)$, and $Y_\pm$ with multiplication by the characteristic function of $R^n_\pm$. The definitions above extend to vector valued functions component-wise.

Let $M(\xi, \eta)$ be an $N \times N$ matrix of functions, positively homogeneous of degree 0, C^{l+1} on $|\xi|^2 + \eta^2 = 1$ where $l > n/2$. The operator $\boldsymbol{M} = F^{-1} M(\xi, \eta) F$ (whose symbol is $M(\xi, \eta)$) is bounded in $H^{s,p}$, invertible (elliptic) if $\det [M(\xi, \eta)] \neq 0$ for $(\xi, \eta) \neq 0$.

THEOREM A. *The operator* $\tilde{\boldsymbol{M}}$: $u \to (Y_- u, Y_+ \boldsymbol{M} u)$ *has a closed range in* $H^{s,p}(R^n_-) \times H^{s,p}(R^n_+)$ *for every s except at most N exceptional values of* s(mod 1). *There exists* $k' \geqq k''$ *such that for* $s = k + \sigma$ *nonexceptional*

$$\|u\|_{s,p} \leqq C[\|Y_- u\|_{s,p} + \|Y_+ \boldsymbol{M} u\|_{s,p}], \quad \text{all } u \in H^{s,p};\ k \geqq k',$$

$$\sum\nolimits_\pm \|V_\pm\|_{-s,p'} \leqq C\|V_- + \boldsymbol{M} V_+\|_{-s,p'}, \quad \text{all } V_\pm \in H^{-s,p'}_\pm\ k \leqq k'';$$

$k'' = k'$ *in the scalar case.*

The first estimate means that $\tilde{\boldsymbol{M}}$ is 1-1 and has a closed range. The second ("dual") estimate assures that the range of $\tilde{\boldsymbol{M}}$ is full. Thus as $s \to +\infty$ the operator $\tilde{\boldsymbol{M}}$ becomes left invertible, as $s \to -\infty$ it becomes right invertible.

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