FAVARD CLASSES FOR n-DIMENSIONAL SINGULAR INTEGRALS

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Let $f \in L_p(E^n)$, $1 \leqq p < \infty$, E^n being the n-dimensional Euclidean space, and let us consider an approximation process defined by means of a singular integral of Fourier convolution type

$$K(f; x; \zeta) = \frac{1}{(2\pi)^{n/2}} \int_{E^n} f(x-u) k(u; \zeta)\, du. \tag{1}$$

Here u, x denote vectors of E^n and ζ a positive parameter, whereas $k(u; \zeta)$ is said to be the kernel of the integral (1) subject to the following conditions [1, p. 1]:

$$\text{(i)} \quad \|k(\cdot; \zeta)\|_1 \leqq M, \qquad \int_{E^n} k(u; \zeta)\, du = (2\pi)^{n/2} \quad \text{for all } \zeta > 0;$$

$$\text{(ii)} \quad \lim_{\zeta \to \infty} \int_{|u| \geqq \delta} |\, k(u; \zeta)\,|\, du = 0 \qquad \text{for all } \delta > 0.$$

It is well known [1, p. 10] that under these conditions the singular integral (1) exists a.e., again belongs to $L_p(E^n)$ and satisfies the relations $\|K(f; \cdot; \zeta)\|_p \leqq \|k(\cdot; \zeta)\|_1 \|f(\cdot)\|_p$ and

$$\lim_{\zeta \to \infty} \|K(f; \cdot; \zeta) - f(\cdot)\|_p = 0. \tag{2}$$

Starting with relation (2) one wishes to establish some connections between the rapidity of the convergence in (2) and further properties of the function f. Here we only aim to discuss a special but nevertheless important case of this general approximation problem, namely the case of the best possible rate of approximation of nontrivial functions f by the singular integral (1), and to determine the exact class F of functions f for which this optimal rate is precisely attained. This notion, the so-called saturation of the process (1), was first introduced by J. Favard [6] and is, for our situation, given by

DEFINITION. *Let the singular integral* (1) *be given with kernel* $k(x; \zeta)$ *and* $f \in L_p(E^n)$, $1 \leqq p < \infty$. *If there exists a monotone decreasing*

[1] The results of this paper were announced by R. J. Nessel in talks held on September 15, 1964 at the Austrian Mathematical Congress, Graz, and on March 6, 1964 and August 5, 1965 at the Mathematical Research Institute, Oberwolfach, Black Forest.