SINGULAR INTEGRALS

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In this paper I will attempt to describe the subject as it has developed in the last fifteen years, outlining the methods by which its problems have been approached and discussing its connections with other branches of Analysis. It will perhaps be best to start by considering certain classical situations which lead naturally to singular integrals and which contain the seeds of some of the methods and some of the applications we will discuss later. The simplest one arises in attempting to establish the connection between the real and imaginary parts of the boundary values of an analytic function. Suppose that $f(z)$ is analytic in $I(z) \geqq 0$ and that $zf(z)$ is bounded, then if $u(x)$ and $v(x)$ are the real and imaginary parts of f on $R(z)=0$, we have

$$f(z) = \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{u(t)}{t - z}\, dt, \qquad v(x) = \lim_{\epsilon\to 0} \frac{1}{\pi}\int_{|x-t|>\epsilon} \frac{u(t)}{x - t}\, dt$$

the second formula being obtained from the first by letting the imaginary part of z tend to zero. The expression for v above is the so-called Hilbert transform of the function u. The integral it involves is absolutely divergent due to the singularity of the kernel at $x=t$.

A second example is given by the second order derivatives of the Newtonian potential. Let us consider the potential of the density function $f(x)$ in R^3

$$g(x) = \frac{-1}{4\pi}\int \frac{f(y)}{|x - y|}\, dy,$$

where $|x-y|$ denotes the distance between the points x and y and dy denotes the volume element in R^3. If f is sufficiently smooth, differentiation under the integral sign leads to the following expression for the second derivatives $D_{ij}g$ of g

$$D_{ij}g(x) = \frac{-1}{4\pi}\int \frac{1}{|x - y|} D_{ij}f(y)\, dy$$

and integration by parts gives

$$D_{ij}g(x) = \tfrac{1}{3}\delta_{ij}f(x) + \lim_{\epsilon\to 0}\int_{|x-y|>\epsilon} k_{ij}(x - y)f(y)\, dy, \tag{1}$$

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