WHITEHEAD TORSION

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In 1935, Reidemeister, Franz and de Rham introduced the concept of "torsion" for certain finite simplicial complexes. For example let X be a finite complex whose fundamental group $\pi_1 X$ is cyclic of order m. We can identify $\pi_1 X$ with the group of covering transformations of the universal covering complex $\hat{X}$. If $\pi_1 X$ operates trivially on the rational homology $H_*(\hat{X}; \mathbf{Q})$, then the *torsion* of X is defined as a certain collection of elements in the algebraic number field $\mathbf{Q}[\exp(2\pi i/m)]$. This torsion is a kind of determinant which describes the way in which the simplexes of $\hat{X}$ are fitted together with respect to the action of $\pi_1 X$. The actual definition will be given in §8. (See Franz [1935], de Rham [1950], Milnor [1961].)

In 1950, J. H. C. Whitehead defined the "torsion" of a homotopy equivalence between finite complexes. This is a direct generalization of the Reidemeister, Franz, and de Rham concept; but is a more delicate invariant. (See §7.) It has gradually been realized that the Whitehead torsion provides a key tool for the study of combinatorial or differentiable manifolds with nontrivial fundamental group. Closely related is the concept of "simple homotopy type" (Whitehead [1939], [1950]).

The Whitehead torsion is not defined as an algebraic number, but rather as an element of a certain commutative group $\mathrm{Wh}(\pi_1 X)$ which depends on the fundamental group. For many years these "Whitehead groups" were completely impossible to compute, except in a very few special cases (Higman [1940]). But recent work by H. Bass and others has made these groups moderately accessible.

The first six chapters of this presentation will be concerned with the algebraic part of the theory, and the remaining six chapters with the geometric applications. There are two appendices: one to show the relationship of Whitehead groups to the work of Bass and Mennicke on congruence subgroups, and one to help motivate the notation $K_1 A$ which is used for the Whitehead group of a ring.

1. The Whitehead group $K_1 A$ of a ring. Let A be an associative ring with unit. The group of all nonsingular $n \times n$ matrices over A will be denoted by $\mathrm{GL}(n, A)$. Identifying each $M \in \mathrm{GL}(n, A)$ with the matrix

$$\begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}(n+1, A)$$