ON HOMOGENEOUS ALGEBRAS[1,2]

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In this report we will sketch a new approach to the theory of finite dimensional nonassociative algebras. Most of the results mentioned here are proved in a joint book with H. Braun [1]. Its intention is to give a satisfying theory for both Jordan algebras and alternative algebras without dealing too much with identities.

1. For simplicity, only finite dimensional algebras $\mathfrak{A}$ with unit element e over a field Φ of characteristic not 2 shall be considered. For $u \in \mathfrak{A}$ we define the powers of u by $u^{m+1} = uu^m$. The left resp. right regular representation $L(u)$ resp. $R(u)$ are given by

$$uv = L(u)v = R(v)u.$$

Besides $\mathfrak{A}$ we consider the commutative algebra $\mathfrak{A}^+$ defined in the same vector space as $\mathfrak{A}$ with the product $u \circ v = (uv + vu)/2$. The left regular representation of $\mathfrak{A}^+$ is $L^+(u) = [L(u) + R(u)]/2$.

By a field extension of a vector space or an algebra $\mathfrak{A}$ we mean any tensor product $\Phi' \otimes_\Phi \mathfrak{A}$, where Φ' is an extension field of Φ. Let $b_1, b_2, \cdots, b_n$ be a basis of $\mathfrak{A}$ over Φ and let be $\tau_1, \tau_2, \cdots, \tau_n$ elements algebraically independent over Φ. Putting $\tilde{\Phi} = \Phi(\tau_1, \tau_2, \cdots, \tau_n)$ we denote by $\tilde{X} = \tilde{\Phi} \otimes_\Phi X$ the vector space obtained from the vector space X by extending Φ to $\tilde{\Phi}$. The element $x = \tau_1 b_1 + \tau_2 b_2 + \cdots + \tau_n b_n$ of $\tilde{\mathfrak{A}}$ is called a *generic element* of $\mathfrak{A}$. Let X be a vector space over Φ and let f be an arbitrary element in $\tilde{X}$. Then f can be regarded as a rational function $f(x)$ of x, because the components of f with respect to a basis of X over Φ are rational in $\tau_1, \tau_2, \cdots, \tau_n$. Hence $f(x + \tau u)$, $u \in \mathfrak{A}$, is rational in the variable τ. Since x is a generic element, the differential operator

$$\Delta_x^u f(x) = \frac{d}{d\tau} f(x + \tau u)\bigg|_{\tau=0}$$

is well defined. Moreover it is linear in u. This operator satisfies the usual rules for a differential operator.

2. Let $\mathfrak{A}$ be a finite dimensional algebra over the field Φ with unit element e and let x be a generic element of $\mathfrak{A}$. We call $\mathfrak{A}$ an *algebra with inverse* if there exists an element $x^{-1} \in \tilde{\mathfrak{A}}$ such that

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[2] An address by Professor Koecher delivered before the meeting of the Society at Massachusetts Institute of Technology on October 30, 1965 by invitation of the Committee to Select Hour Speakers for Eastern Sectional Meetings; received by the editors November 29, 1965.