ON A FIXED POINT THEOREM FOR NONLINEAR P-COMPACT OPERATORS IN BANACH SPACE[1]

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1. **Introduction.** In [5] Kaniel proved a fixed point theorem for a nonlinear quasi-compact operator in a Banach space. The purpose of this paper is to generalize and simplify Kaniel's main result and its proof to a more general class of nonlinear operators which we call projectionally-compact (P-compact) and which, among others, contains completely continuous, quasi-compact, and monotone operators. From our general fixed point theorem for P-compact operators we then deduce in a simple way the fixed point theorems of Schauder [11], Rothe [10], Krasnoselsky [6], Altman [1], Kaniel [5], and others. In case the underlying space is a Hilbert space, we deduce (see also Kaniel [5]) some theorems on strongly monotone operators obtained by Minty [7] and Browder [2], [3], [4]. Let us add that our conditions have a form which not only admits a simpler investigation but at the same time seem to be more natural and suitable for applications to numerical functional analysis. Furthermore, the method of our proof is basically constructive. In fact, we show in [9] that it is essentially the projection method of which the Galerkin method is one of its simplest realizations. The latter methods are known [8] to play an important role in the approximate solution of operator equations.

2. **Preliminary results.** Let X be a finite dimensional Banach space; let B_r denote the closed ball in X of radius $r>0$ about the origin and let S_r denote the boundary of B_r. For later use we state first an essentially known fixed point theorem whose brief proof, which is based on the Brouwer fixed point theorem and the retraction mapping principle is given in [9].

THEOREM 1. *Let A be a continuous mapping of B_r into X and let μ be any constant. Then there exists at least one element u in $B_r - S_r$ such that*

$$Au - \mu u = 0 \tag{1}$$

provided that the mapping A satisfies the condition:

(π_μ): *If for some x in S_r the equation $Ax = \alpha x$ holds then $\alpha < \mu$.*

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[1] The expanded version of this paper with detailed proofs will appear in [9].