A COMPLETE EXTREMAL DISTANCE PROBLEM ON OPEN RIEMANN SURFACES[1]

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Partition the boundary contours of a compact bordered Riemann surface $\overline{W}$ into four disjoint sets α_0, α, β, γ with α_0 and α nonempty. Let F consist of all arcs in $\overline{W}-\gamma$ which have initial point in α_0 and endpoint in α. Let F^* consist of all cycles in $\overline{W}-\beta$ which separate α_0 from α. Determine the harmonic function u in $\overline{W}$ by the boundary conditions $u=0$ on α_0, $u=1$ on α, $\partial u/\partial n=0$ along γ, and u is constant on each contour β_i in β with the constant so chosen that $\int_{\beta_i} du^*=0$. Then $\lambda(F)=\lambda^{-1}(F^*)=\|du\|^{-2}$ where $\lambda(\cdot)$ denotes the extremal length and $\|\cdot\|^2$ denotes the Dirichlet norm. This result is implicit in the fundamental work of Ahlfors-Beurling [1]. Observe that if $\overline{W}$ is planar and α_0, α are single contours then $\exp 2\pi(u+iu^*)/\|du\|^2$ is a conformal map of Int $\overline{W}$ into $1<|z|<\exp 2\pi/\|du\|^2$, the components of β going onto circular slits and the components in γ onto radial slits.

The purpose of this note is to announce a complete generalization of the above result which is valid for arbitrary open Riemann surfaces. As a consequence of our work we obtain a new class of conformal mappings of plane regions onto "extremal" slit annuli analogous to the situation described above. These results and their proofs will be published in a forthcoming paper [2].

We begin with an open Riemann surface W and partition its Kerékjártó-Stoilöw ideal boundary into four disjoint sets α_0, α, β, γ with α_0 and α nonempty. For technical reasons we assume that α_0, α, $\alpha_0\cup\alpha\cup\beta$ are closed subsets of the Kerékjártó-Stoilöw compactification $\hat{W}$ of W. Let $\mathfrak{F}$ be the family of arcs in $\hat{W}-\gamma$ with initial points in α_0 and end points in α. Let $\mathfrak{F}^*$ consist of all suitably orientated τ such that τ is a countable union of closed curves in $\hat{W}-\alpha_0-\alpha-\beta$, all limit points of τ are contained in γ, and no component of $\hat{W}-\gamma-\tau$ contains points in both α_0 and α. There is a natural definition for $\lambda(\mathfrak{F})$, $\lambda(\mathfrak{F}^*)$ obtained by replacing each curve $\tau\subset\hat{W}$ by the curve $\tau\cap W$. An HD-function u on W is constructed which generalizes the u defined above for compact bordered surfaces. The actual definition of u uses a noncompact exhaustion of W "in the direction

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