MEASURES OF AXIAL SYMMETRY FOR OVALS[1]

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B. Grünbaum [2] has made a thorough report of the known results on measures of central symmetry for convex sets. We seek here to measure the degree of axial symmetry (axiality) of an oval K (a compact convex set in E^2 with interior points).

DEFINITION. A measure of axiality is a real-valued function f defined on the class of ovals such that

(i) $0 \leqq f(K) \leqq 1$;

(ii) $f(K) = 1$ if and only if K has an axis of symmetry (is axial);

(iii) f is similarity-invariant.

Let ϕ be a direction in the plane, $k(\phi)$ a line normal to the direction ϕ, $b_\phi(K)$ the breadth of K in the direction ϕ, $\mathrm{Cv}(S)$ the convex hull of the set S, $\lambda_\phi(K)$ the "load curve" of K in the direction ϕ, (i.e., the set of midpoints of all chords of K in the direction ϕ), $[K]$ the area of K, $|K|$ the perimeter of K, and $K_{k(\phi)}$ the Steiner symmetrand of K with respect to the line $k(\phi)$.

The following measures of axiality are studied, and lower bounds are determined for them on the classes of arbitrary ovals (K), centrally symmetric ovals (K_c), and ovals of constant breadth (K_1):

$$f_1(K) = \max_\phi \{1 - b_\phi[\mathrm{Cv}(\lambda_\phi(K)]/b_\phi(K)\},$$

$$f_2(K) = \max_\phi \max_k (1/b) \int_0^b r(\phi, k, y)\, dy,$$

where $b = b_{\phi+\pi/2}(K)$, $k = k(\phi)$, and $r(\phi, k, y)$ is the ratio (taken $\leqq 1$) of the lengths of the two parts into which a chord $\gamma = \gamma(y)$ of K in the direction ϕ is divided by k ($r = 0$ if $\gamma \cap k = \varnothing$),

$$f_3(K) = \max_{K'} \{[K']/[K]: K' \text{ is axial, and } K' \subseteq K\},$$

$$f_4(K) = \max_{K''} \{[K]/[K'']: K'' \text{ is axial, and } K \subseteq K''\},$$

$$f_5(K) = \max_{K'} \{|K'|/|K| : K' \text{ is axial, and } K' \subseteq K\},$$

$$f_6(K) = \max_{K''} \{|K|/|K''| : K'' \text{ is axial, and } K \subseteq K''\},$$

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