FIRST ORDER PROPERTIES OF PAIRS OF CARDINALS

BY H. JEROME KEISLER

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We consider models of a countable first order logic L with an identity symbol and predicate symbols U, P_0, P_1, $\cdots$, U being unary. A model $\mathfrak{A} = \langle A, U_{\mathfrak{A}}, P_{0\mathfrak{A}}, \cdots \rangle$ for L is said to be a *two-cardinal model* if A is infinite and the power of $U_{\mathfrak{A}}$ is less than the power of A. By a *set of axioms for two-cardinal models* we mean a set Σ of sentences of L such that $\mathfrak{A}$ is a model of Σ if and only if there exists a two-cardinal model which is elementarily equivalent to $\mathfrak{A}$. Using results of Fuhrken [1], Vaught [4] proved the following theorem.

THEOREM (VAUGHT). *There is a set of axioms for two-cardinal models. If the language L is recursive, then there is a recursive set of axioms for two-cardinal models.*

We say that L is recursive if the number of argument places of the symbol P_n is a recursive function of n. Vaught's proof depends on the fact that if Σ^* is a recursive set of sentences in an extension L^* of the language L, then there is a recursive set Σ of sentences of L such that Σ and Σ^* have exactly the same consequences in L. In principle his proof can be used to construct a particular set of axioms for two-cardinal models, but the set seems to be so complicated that in practice one cannot easily tell whether or not a given sentence belongs to it. Vaught has proposed the problem of finding a simple set of axioms for two-cardinal models. The author heard about Vaught's problem through Dana Scott.

In this note we shall give a particular set of axioms for two-cardinal models which is simple enough to be written down as a fairly short axiom scheme. Our theorem was stated without proof in [2]. Let the individual variables of L be v_i, x_i, y_i, z_i, where $i = 0, 1, 2, \cdots$.

THEOREM 1. *A set of axioms for two-cardinal models is given by the set Γ of all sentences of the form*

$$(*)\qquad \begin{gathered} \exists v_0 \forall x_0 \exists y_0 z_0 \cdots \forall x_n \exists y_n z_n \\ \left[\bigwedge_{i=0}^{n} v_0 \neq y_i \;\&\; \bigwedge_{i,j=0}^{n} (U(x_j) \;\&\; x_i = z_j \rightarrow y_i = x_j) \right. \\ \left. \&\; \bigwedge_{j=0}^{m} (\phi_j(x_0, \cdots, x_n) \rightarrow \phi_j(y_0, \cdots, y_n)) \right]. \end{gathered}$$