EXAMPLES IN HELSON SETS

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A compact subset P of a locally compact abelian group G is said to be a *Kronecker set* in G [1, p. 97] if every continuous unimodular function on P is uniformly approximable on P by continuous characters of G. P is a *Helson set* [1, pp. 114–115] if for some $\epsilon > 0$ and each $\mu \in M(P)$:

$$(H, \epsilon) \quad \epsilon \|\mu\| \leqq \sup_{\gamma \in \Gamma} \left| \int_G \gamma(x) d\mu(x) \right|, \qquad \|\mu\| = |\mu|(P),$$

Γ being the dual of G.

If P is a Kronecker set in G, P satisfies $(H, 1)$ by [1, Lemma 5.5.1]. It was asked in [1] whether $(H, 1)$ implies that P is a Kronecker set. Wik [2] constructed a class of counter-examples in the real line; in this note a different type of construction is announced.

Let X be a compact Hausdorff space and U the (abstract) group of continuous unimodular functions on X, Γ a subgroup of U which separates the points of X. Then X is embedded as a topological subspace of $\hat{\Gamma}$ and is a Kronecker set in $\hat{\Gamma}$ if and only if Γ is uniformly dense in U. We give below two examples in which Γ is a proper closed subgroup of U but for which $(H, 1)$ holds for measures in X.

(a) X is the 1-torus and Γ the group of functions with winding number, or degree, zero. In this case the Kronecker condition holds on the complement of any arc, so $(H, 1)$ holds.

(b) X is the unit interval $[0, 1]$ and Γ is the set of all functions e^{if}, f real and $\int_0^1 f\,dx = 0$. In this case $U = \Gamma \cdot \mathbf{C}$, $\mathbf{C}$ being the subgroup of constant functions.

In (a) and (b) the groups Γ have the form $\exp^{iH}$, where H is an additive subgroup of the real continuous functions on X. In each case H contains a dense subgroup H_1 algebraically isomorphic to $Z \oplus Z \oplus Z \oplus \cdots$; the exponential mapping is an isomorphism onto Γ. In (a) H_1 is the subgroup of trigonometric polynomials with coefficients in $Z + \sqrt{2}Z$; in (b) one uses the same coefficients with the generators $\{x^n - 1/(n+1) : n \geqq 1\}$. Using the smaller subgroups of U determined by these subspaces we can embed $X \to T^\omega$ and have the same phenomenon in regard to measures in X. In view of Theorem