HIGHER PRODUCTS

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W. S. Massey has defined a class of higher order cohomology operations of several variables, the higher products [2]. In this paper, we shall present a relativized definition of the higher products. We shall go on to list some of the algebraic and functorial properties of these operations. Finally, we shall describe a related cohomology operation of one variable. In certain cases, the latter operation can be computed in terms of primary Steenrod operations.

1. **Notation and definitions.** Throughout this paper, let $\overline{X}$ be a topological space and let (X_i, A_i) be pairs of subspaces of $\overline{X}$, for $i=1, \cdots, k$, such that $\bigcup_{r=1}^{k} A_r \subset \bigcap_{r=1}^{k} X_r$. Furthermore, for $1 \leqq i, j \leqq k$, assume that the triads $(\overline{X}, A_i, A_j)$ are excisive in the singular cohomology theory. This condition is satisfied if each X_i and A_i are open in $\overline{X}$ or if $\overline{X}$ is a CW complex and each X_i and A_i are subcomplexes. Let $u_1, \cdots, u_k$ be cohomology classes in the singular cohomology groups $H^{p_1}(X_1, A_1), \cdots, H^{p_k}(X_k, A_k)$ respectively, where the coefficients are in a fixed commutative ring R with identity. Finally, let $p(i, j) = \sum_{r=i}^{j} p_r - 1$ and $(X, A) = (\bigcap_{r=1}^{k} X_r, \bigcup_{r=1}^{k} A_r)$.

Under certain conditions, we may define the k-fold product $\langle u_1, \cdots, u_k \rangle$. Our definition shall be similar to the provisional definition of Massey [2].

DEFINITION 1. A defining system for $\langle u_1, \cdots, u_k \rangle$, A, is a set of singular cochains $(a_{i,j})$, for $1 \leqq i \leqq j \leqq k$ and $(i, j) \neq (1, k)$, satisfying the conditions:

(1.1) $a_{i,j} \in C^{p(i,j)+1}(\bigcap_{r=i}^{j} X_r, \bigcup_{r=i}^{r} A_r)$,

(1.2) $a_{i,i}$ is a cocycle representative of u_i, $i=1, \cdots, k$ and

(1.3) $\delta a_{i,j} = \sum_{r=i}^{j-1} (-1)^{(j+1-r)p(i,r)} a_{i,r}\, a_{r+1,j}$.

The related cocycle of A is the singular cocycle of $C^*(X, A)$

(1.4) $\sum_{r=1}^{k-1} (-1)^{(k+1-r)p(1,r)} a_{1,r}\, a_{r+1,k}$.

DEFINITION 2. The k-fold product $\langle u_1, \cdots, u_k \rangle$ is said to be defined if there is a defining system for it. If it is defined, then $\langle u_1, \cdots, u_k \rangle$ consists of all classes $w \in H^{p(1,k)+2}(X, A)$ for which there exists a defining system A whose related cocycle represents w.

If $k=2$, then the higher product $\langle u_1, u_2 \rangle$ is the ordinary cup product

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