METARECURSIVELY ENUMERABLE SETS AND ADMISSIBLE ORDINALS

BY GERALD E. SACKS[1]

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Here we describe some results which will be proved in detail in [13] and [14]. The notion of metarecursive set was introduced in [8]. Kreisel [7] reported on some of the model-theoretic deliberations which preceded the definitions of [8]. Metarecursion theory is a generalization of ordinary recursion theory from the natural numbers to the recursive ordinals. Theorems about finite sets of natural numbers are replaced by theorems about metafinite sets of recursive ordinals, some of which are infinite. Initially, metarecursive sets were defined in [8] in terms of hyperarithmetic sets, Π_1^1 sets, and notations for recursive ordinals [6], [16]; however, it later proved convenient to utilize an equation calculus devised by Kripke [9]. The purpose of Kripke's theory is to generalize recursion theory from the natural numbers to certain initial segments of the ordinals [9], [10], [11]. He calls an ordinal α admissible if the ordinals less than α have certain closure properties definable in terms of an equation calculus modeled on Kleene's. Kripke's equation calculus has numerals denoting ordinals, finitary substitution rules, and one infinitary deduction rule. If an ordinal α is admissible, then an α-recursive function f is defined by a finite system of equations: each value of f is computable using Kripke's rules, and only correct values can be so computed. It turned out that the first admissible ordinal after ω was Kleene's ω_1, the least nonrecursive ordinal, and that the metarecursive functions were the same as the ω_1-recursive functions [8], [9].

In this paper we concentrate on our first love, metarecursion theory, but we cannot resist noting, whenever appropriate, which of our results generalize to arbitrary admissible ordinals.

A set of recursive ordinals is called *regular* if its intersection with every metafinite set of recursive ordinals is metafinite. (The metafinite sets coincide with the bounded, metarecursive sets.) It was observed in [8] that there exist bounded, metarecursively enumerable sets which are not metarecursive; each such set is a constructive example of a nonregular set. It would not be unfair to say that the interesting arguments of metarecursion theory, if it is granted that

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