UNIFORM ASYMPTOTIC EXPANSIONS OF THE MODIFIED BESSEL FUNCTION OF THE THIRD KIND OF LARGE IMAGINARY ORDER[1]

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The modified Bessel equation

$$\frac{d^2w}{dz^2} + \frac{1}{z}\frac{dw}{dz} - \left(1 - \frac{\nu^2}{z^2}\right)w = 0 \tag{1}$$

with its particular solution $w = K_{i\nu}(z)$, the modified Bessel function of the third kind with pure imaginary order is of fundamental significance in the diffraction theory of pulses. Moreover this function is the kernel of the Lebedev transform [3].

With the exception of Friedlander's results [2] little information is available about the behavior of $K_{i\nu}(z)$ when both ν and z are large which case is of great importance in the applications. In [2] Langer's differential equation method is applied and an asymptotic formula is given for the function and for the zeros of its derivative.

The aim of this paper is to give a fairly complete description of $K_{i\nu}(z)$ for $\nu \to \infty$; the proofs will be given elsewhere.

Based on (1) and on Olver's Theorem B [4], [6] a uniform asymptotic series is constructed in terms of the Airy function $Ai(\xi)$ and of its derivative $Ai'(\xi)$ for $\nu \to \infty$ in a region R which contains the sector $\operatorname{Re} z \geqq 0$, $z \neq 0$.

$$\begin{aligned} K_{i\nu}(\nu z) = \frac{\pi 2^{1/2}}{\nu^{1/3}} \exp\left(-\frac{\pi}{2}\nu\right)\left(\frac{\zeta}{z^2-1}\right)^{1/4} \Bigg\{ & Ai(\xi)\left[1 + \sum_{s=1}^{m} \frac{A_s(\zeta)}{\nu^{2s}}\right] \\ & + \frac{Ai'(\xi)}{\nu^{4/3}} \sum_{s=0}^{m-1} \frac{B_s(\zeta)}{\nu^{2s}} + \frac{\exp\{-\frac{2}{3}\xi^{3/2}\}}{1 + |\xi|^{1/4}} \cdot O(\nu^{-2m-1}) \Bigg\}, \end{aligned} \tag{2}$$

where $\xi = \nu^{2/3}\zeta$, $\frac{2}{3}\zeta^{3/2} = (z^2-1)^{1/2} - \operatorname{arcsec} z$, and the coefficients are given by

$$A_s(\zeta) = \sum_{m=0}^{2s} (-1)^m b_m \zeta^{-3m/2} U_{2s-m}, \quad \zeta^{1/2} B_s(\zeta) = \sum_{m=0}^{2s+1} (-1)^m a_m \zeta^{-3m/2} U_{2s-m+1},$$

with

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