A SIMPLE PROOF OF THE RABIN-KEISLER THEOREM

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For terminology and notation we refer to the two relevant papers of Rabin [3] and Keisler [1]. The following theorem is proved in [1] and is an improvement of the main result of [3].

THEOREM (RABIN-KEISLER). *Let α be an infinite nonmeasurable cardinal. Then every model of power α has a proper elementary extension of the same power if and only if $\alpha = \alpha^\omega$.*

The simple proof referred to in the title does not require the elaborate apparatus of limit ultrapowers (see [1]) or the generalized continuum hypothesis and that α be accessible (see [3]). On the other hand, the proof owes much to certain ideas in [3] and Keisler [2].

One direction of the theorem follows easily from elementary properties of ultrapowers. The following lemma will establish the other direction.

LEMMA. *Suppose α is an infinite nonmeasurable cardinal, $\mathfrak{M} = \langle A, R, S, \cdots \rangle$ is the complete model over a set A of power α, and $\mathfrak{M}' = \langle A', R', S', \cdots \rangle$ is a proper elementary extension of $\mathfrak{M}$. Then $|A'| \geqq \alpha^\omega$.*

PROOF. By a well-known result in set theory (using finite sequences of elements from A), there exists a family

$$P = \{P_\beta : \beta < \alpha^\omega\}$$

of countably infinite subsets P_β of A such that $|P| = \alpha^\omega$ and $P_\beta \cap P_\gamma$ is finite whenever $\beta \neq \gamma$. Well-order each P_β,

$$P_\beta = \{p_{\beta n} : n < \omega\}.$$

Let $x \in A' - A$, and let

$$D = \{Q : Q \subset A \text{ and } x \in Q'\}.$$

It is easily seen that D is a nonprincipal ultrafilter over A. By hypothesis D is countably incomplete. Hence, there exists a strictly decreasing sequence

$$A = Q_0 \supset Q_1 \supset \cdots \supset Q_n \supset \cdots$$

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