PIECEWISE LINEAR NORMAL MICRO-BUNDLES

BY C. T. C. WALL

Communicated by J. Milnor, March 4, 1965

The object of this paper is to gain some information about the unstable piecewise linear groups. The tool that we use for this purpose is the s-cobordism theorem (which has been established for piecewise linear manifolds by J. Stallings [9] and D. Barden [1]). All manifolds and micro-bundles in this paper are piecewise linear, unless otherwise specified.

THEOREM 1. *Let M^m be a compact manifold such that $\pi_1(\partial M)\cong\pi_1 M$ by inclusion, and let $f: K^k\to M^m$ be a simple homotopy equivalence of a finite simplicial k-complex with M. Then if $m\geqq 6$, $m\geqq 2k+1$, there is a compact manifold L such that $\pi_1(\partial L)\cong\pi_1 L$, and $L\times I\cong M$. If $m\geqq 7$, $m\geqq 2k+2$, L is uniquely determined.*

PROOF. We first observe that the pair $(M, \partial M)$ is $(m-k-1)$-connected. Indeed, since inclusion induces an isomorphism of fundamental groups, we can use the relative Hurewicz theorem to compute the first nonvanishing relative homotopy group: $\pi_i(M, \partial M)\cong\pi_i(\tilde{M}, \partial\tilde{M})$ (where $\tilde{M}$ denotes the universal cover), $\pi_i(\tilde{M}, \partial\tilde{M})\cong H_i(\tilde{M}, \partial\tilde{M})\cong H_c^{m-i}(\tilde{M})$, by duality, where c denotes compact cohomology, and $H_c^{m-i}(\tilde{M})\cong H_c^{m-i}(\tilde{K})$ vanishes for $i<m-k$.

It follows that for $m\geqq 2k+1$, f is homotopic to a map $g: K\to\partial M$. If, now $m\geqq 2k+2$ we can move g into general position (see e.g. [11, Chapter 6, Theorem 18]) and so suppose it an imbedding. Take a regular neighbourhood L of $g(K)$ in ∂M. Then L is a manifold, and the inclusion $L\subset M$ is a simple homotopy equivalence.

If $m=2k+1$, g will in general have singularities, transverse selfintersections of k-simplexes of K. For each such selfintersection $Q=g(P_1)=g(P_2)$, we join P_1 to P_2 by a path α in K such that $g(\alpha)$ is a nullhomotopic loop (since $g_*: \pi_1(K)\to\pi_1(\partial M)$ is onto, this is possible). As $k\geqq 3$, we can now map a disc D^2 into ∂M, with its interior imbedded, and meeting $g(K)$ only in its boundary, which is attached along $g(\alpha)$. Proceeding thus for each selfintersection Q, we obtain an imbedding of a complex K' simply homotopy-equivalent to K; we can then take a regular neighbourhood to obtain L, as above. Note in either case that as regular neighbourhood of a subcomplex K' of codimension $\geqq 3$, L has the property $\pi_1(\partial L)\cong\pi_1 L$, for ∂L is a deformation retract of $L-K'$.