A CONJECTURE OF J. NAGATA ON DIMENSION AND METRIZATION[1]

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THEOREM 1. *A metrizable space X is of dimension $\leqq n$ if and only if X admits a metric compatible with the topology which satisfies the condition (n): For any $n+3$ points $x, y_1, \cdots, y_{n+2}$ of X there exist distinct indices i, j such that $d(y_i, y_j) \leqq d(x, y_j)$.*

In this paper we outline briefly a proof of Theorem 1, which was conjectured by J. Nagata [1].

By dimension we shall always mean covering dimension. A family of subsets of X is discrete if each point of X has a neighborhood which meets at most one member of the family. For a subset A of X and a family $\mathcal{C}$ of subsets of X, let $S(A, \mathcal{C})$ denote the union of A and all those $C \in \mathcal{C}$ such that $C \cap A \neq \emptyset$. For each integer $n \geqq 0$, let

$$S^n(A, \mathcal{C}) = \begin{cases} A & \text{if } n = 0, \\ S(S^{n-1}(A, \mathcal{C}), \mathcal{C}) & \text{if } n > 0; \end{cases}$$

$$[\mathcal{C}]^n = \{S^n(C, \mathcal{C}) : C \in \mathcal{C}\}.$$

Let X be a metrizable space of dimension $\leqq n$. For each positive integer j there exist $n+1$ discrete families of open sets, $\mathcal{U}_j^1, \mathcal{U}_j^2, \cdots, \mathcal{U}_j^{n+1}$ such that if $\mathcal{U}_j = \bigcup_{i=1}^{n+1} \mathcal{U}_j^i$, then:

(1) each $\mathcal{U}_j$ covers X;

(2) for each $x \in X$, $\{S(x, \mathcal{U}_j) : j = 1, 2, \cdots\}$ is a neighborhood base at x;

(3) $[\mathcal{U}_{j+1}]^{31}$ refines $\mathcal{U}_j$ for each j;

(4) if $j < k$ and $1 \leqq i \leqq n+1$, each member of $[\mathcal{U}_k]^{31}$ meets at most one member of $\mathcal{U}_j^i$.

The $\mathcal{U}_j^i$ are defined inductively on j. Their construction relies on a new characterization of dimension [2].

THEOREM 2. *A metrizable space X is of dimension $\leqq n$ if and only if for each open cover $\mathcal{C}$ of X there exist $n+1$ discrete families of open sets, $\mathcal{U}^1, \mathcal{U}^2, \cdots, \mathcal{U}^{n+1}$ such that $\bigcup_{i=1}^{n+1} \mathcal{U}^i$ is a cover of X which refines $\mathcal{C}$.*

PROOF OF THEOREM 1. Let R^* denote the set of dyadic rationals in the open interval $(0, 1)$. For each $m \in R^*$ there exist $n+1$ discrete

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