SEMI-REGULAR MAXIMAL ABELIAN SUBALGEBRAS IN HYPERFINITE FACTORS[1]

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A factor is a ring of operators whose center consists only of scalar multiples of the identity. Murray and von Neumann have defined various kinds of factors, calling a continuous factor with finite trace a type II_1 factor [3], [4]. Dixmier began the detailed study of maximal abelian subalgebras of type II_1 factors. He defined regular, semi-regular but not regular, and singular maximal abelian subalgebras, and showed that at least one of each type exists [2]. His II_1 factors turn out to be hyperfinite in algebraic type. The factors we consider are also hyperfinite. In this note we discuss their semi-regular subalgebras, and present an isomorphism invariant which allows us to obtain new existence results.

Let $\mathfrak{A}$ be a hyperfinite factor, $\boldsymbol{R}$ a maximal abelian subalgebra of $\mathfrak{A}$. For any subring $\boldsymbol{D}$ of $\mathfrak{A}$, $N(\boldsymbol{D})$ is the ring generated by all unitaries which leave $\boldsymbol{D}$ invariant, and $N^k(\boldsymbol{D})=N[N^{k-1}(\boldsymbol{D})]$. In particular, we let $N(\boldsymbol{R})=\boldsymbol{P}$. $\boldsymbol{R}$ is semi-regular but not regular iff $\boldsymbol{P}$ is a factor not equal to $\mathfrak{A}$. In [5] we defined an isomorphism invariant for such subalgebras, which we called *length*. If $\boldsymbol{R}\subset\boldsymbol{P}\subset N(\boldsymbol{P})\subset\cdots\subset N^L(\boldsymbol{P})=\mathfrak{A}$, (when $\boldsymbol{R}\neq\boldsymbol{P}\neq N(\boldsymbol{P})\neq\cdots\neq N^L(\boldsymbol{P})$) then L is the length of $\boldsymbol{R}$. By constructing a semi-regular subalgebra $\boldsymbol{R}$ of every length $L=1, 2, 3, \cdots$, we obtained an infinite sequence of subalgebras which could not be pairwise connected by *-automorphisms of $\mathfrak{A}$.

Another possible invariant is *product type.* Suppose $\boldsymbol{R}$ has length L. Then $\boldsymbol{R}$ is of product type α, $0\leqq\alpha\leqq L$, iff the following holds: For every t, $1\leqq t\leqq\alpha$, there exist S_1 and S_2 in $N^{t-1}(\boldsymbol{P})^{\perp}\cap N^t(\boldsymbol{P})$ such that the product $S_1S_2\neq 0$ is in $N^{t-1}(\boldsymbol{P})^{\perp}\cap N^t(\boldsymbol{P})$. But for s such that $\alpha\leqq s\leqq L$, every T_1 and T_2 in $N^{s-1}(\boldsymbol{P})^{\perp}\cap N^s(\boldsymbol{P})$ have their product T_1T_2 in $N^{s-1}(\boldsymbol{P})$. (Taking of orthogonal complements is meaningful, for within a II_1 factor, the weak, strong, and Hilbert space (metric) closures of a subalgebra all coincide [4]. The metric topology is based on the norm derived from the scalar product $(A, B)=\mathrm{Tr}(B^*A)$ for A, B in $\mathfrak{A}$.)

THEOREM 1. *Suppose $\boldsymbol{R}$ and $\boldsymbol{R}'$ are semi-regular but not regular subalgebras of $\mathfrak{A}$, and $\boldsymbol{R}$ has product type α, while $\boldsymbol{R}'$ has product type*

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