THE VARIANCE OF THE NUMBER OF ZEROS OF A STATIONARY NORMAL PROCESS[1]

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Let $x(t)$ be a separable stationary normal stochastic process with zero mean, covariance function $r(\tau)$ (with $r(0)=1$, for convenience), and spectrum $F(\lambda)$ having an absolutely continuous component. Let N denote the number of times $x(t)$ crosses the zero level in $0\leqq t\leqq T$ and write λ_2 for the second spectral moment $\int_0^\infty \lambda^2\, dF(\lambda) = -r''(0)$. Then it is well known that the mean of the random variable N is given by $T\sqrt{\lambda_2}/\pi$ (and, in fact, it has been recently shown by Ylvisaker [4] that this is true whether or not λ_2 is finite, provided $x(t)$ has continuous sample functions with probability one). The second moment has been given by a number of authors (e.g., [2], [3]) but the best conditions available to date include the existence of a sixth derivative for the covariance function $r(\tau)$.

Here we give the following result for the second moment of N and indicate briefly the general lines of proof. Full details will appear elsewhere.

THEOREM. *Suppose that* $\lambda_2 < \infty$ *and that, for all sufficiently small* τ, $\lambda_2 + r''(\tau) \leqq \Psi(\tau)$, *where* $\Psi(\tau)$ *decreases as* τ *decreases to zero and is such that* $\Psi(\tau)/\tau$ *is integrable over* $[0, T]$. *Then*

$$(1)\quad \mathcal{E}\{N^2\} = \mathcal{E}\{N\} + \int\int_0^T dsdt \int\int_{-\infty}^{\infty} |xy|\, p_{t-s}(0, 0, x, y)\, dxdy,$$

where $p_\tau(u, v, x, y)$ *is the four-variate normal density for* $x(0)$, $x(\tau)$ *and the (mean square) derivatives* $x'(0)$, $x'(\tau)$. *That is,* p_τ *has the normal form with zero means and covariance matrix*

$$\Sigma = \begin{bmatrix} 1 & r(\tau) & 0 & r'(\tau) \\ r(\tau) & 1 & -r'(\tau) & 0 \\ 0 & -r'(\tau) & \lambda_2 & -r''(\tau) \\ r'(\tau) & 0 & -r''(\tau) & \lambda_2 \end{bmatrix}.$$

(1) *can be evaluated explicitly to yield*

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