LIAPUNOV FUNCTIONS AND GLOBAL EXISTENCE

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Liapunov functions have been used to characterize the stability, the asymptotic stability, and the boundedness of solutions of ordinary differential equations. The purpose of this announcement is to characterize the continuability of solutions. Details and applications will appear elsewhere [6].

Consider the equation

$$x' = f(t, x) \qquad \left(' = \frac{d}{dt}\right), \tag{E}$$

where x and f belong to E^n, $f(t, 0)=0$ for $t \geqq 0$, and f is continuous and locally Lipschitzian on D, $D=\{(t, x): t\geqq 0,\ x\in E^n\}$. For $(t_0, x_0) \in D$, let $F(t, t_0, x_0)$ be that solution of (E) for which $F(t_0, t_0, x_0)=x_0$.

DEFINITION 1. *$V(t, x)$ is a Liapunov function for* (E) *if $V(t, x)$ is non-negative, continuous, and locally Lipschitzian on D, if $V(t, 0)=0$ for $t\geqq 0$, and $V'(t, x)\leqq 0$, where*

$$V'(t, x) = \limsup_{h\to 0+} h^{-1}[V(t+h, x+hf(t, x)) - V(t, x)].$$

DEFINITION 2. *$V(t, x)$ is mildly unbounded if for every $T>0$, $V(t, x)\to+\infty$ as $|x|\to\infty$ uniformly in t, $0\leqq t\leqq T$.*

THEOREM. *For f as described, the solution $F(t, t_0, x_0)$ of* (E) *can be continued to $[t_0, \infty)$ for every $(t_0, x_0)\in D$ if and only if there exists on D a mildly unbounded Liapunov function $V(t, x)$ for* (E). *Furthermore, this function is positive definite if and only if the zero solution of* (E) *is stable.*

SKETCH OF PROOF. Assume such a function exists. If for some $(t_0, x_0) \in D$, $F(t, t_0, x_0)$ cannot be continued to $[t_0, \infty)$, there exists $T>t_0$ such that $|F(t, t_0, x_0)|\to\infty$ as $t\to T-0$. Since V is mildly unbounded, $V(s, F(\tau, t_0, x_0))\to+\infty$ as $\tau\to T-0$ uniformly on $0\leqq s\leqq T$, hence, $V(\tau, F(\tau, t_0, x_0))\to+\infty$, contradicting $V'(t, x)\leqq 0$.

Conversely, suppose all solutions can be continued. For each positive integer m, let $\phi_m(x)$ be a real-valued C^1 function on E^n such that

$$\phi_m(x) = \begin{cases} 1 & \text{if } |x| \leqq m, \\ 0 & \text{if } |x| \geqq m+1, \end{cases}$$

and consider

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