HOLOMORPHIC CONVEXITY OF TEICHMÜLLER SPACES[1]

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Let B be the complex Banach space of holomorphic functions $\phi(z)=\phi(x+iy)$ defined for $y<0$, with norm $\|\phi\|=\sup|y^2\phi(z)|$. The universal Teichmüller space T may be considered as a subset of B defined as follows [2], [7]. $\phi\in B$ belongs to T if and only if there is a quasiconformal selfmapping $w(z)$ of the z-plane which leaves 0 and 1 fixed and is, for $y<0$, a conformal mapping with Schwarzian derivative $\phi(z)$. If this is the case we say that w belongs to ϕ. T is a bounded domain in B containing the origin. The so-called Teichmüller metric (see below) is defined in T; it is topologically equivalent to the metric of B. Every boundary point of T has infinite Teichmüller distance from the origin.

If $Q\subset T$, we denote by $h(Q)$ the hull of Q with respect to continuous holomorphic functions in T. $\psi\in T$ belongs to $h(Q)$ if and only if there is no continuous holomorphic function f in T such that $|f(\psi)|>|f(\phi)|$ for all $\phi\in Q$.

THEOREM 1. *If $Q\subset T$ is bounded in the Teichmüller metric, so is $h(Q)$.*

PROOF. For $\phi\in T$ let $K(\phi)$ denote the smallest dilitation of a mapping w belonging to ϕ. The function $K(\phi)$ is well defined and $\log K(\phi)$ is the Teichmüller distance of ϕ to the origin.

For $\phi\in T$ and any three real numbers $a<b<c$ set $f_{a,b,c}(\phi)=(w(b)-w(a))/(w(c)-w(a))$ where w is any mapping belonging to ϕ. These functions are well defined and one verifies, using [3], that they are continuous and holomorphic in T.

Let $\phi\in T$ and $K(\phi)\leqq\alpha$. Then there is a w belonging to ϕ with dilitation not exceeding α. Let Γ be the image of the real axis under w; this curve depends only on ϕ. Set $\chi(\zeta)=w(w^{-1}(\zeta)^*)$ where the asterisk denotes complex conjugation. Then χ is a quasireflection about Γ, that is an orientation-reversing topological selfmapping of the plane which leaves every point of Γ fixed, and the dilitation of χ is at most α^2. By a theorem of Ahlfors [2] it follows that $|f_{a,b,c}(\phi)|\leqq\beta$ for all $a<b<c$, where β depends only on α.

Assume now that $|f_{a,b,c}(\phi)|\leqq\alpha$ for all $a<b<c$ and let Γ be the image of the real axis under a mapping w belonging to ϕ. Again by

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