THE METASTABLE HOMOTOPY OF $O(n)$

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It is not easy to determine how many trivial line bundles can be split off a stable real vector bundle; the first crucial question concerns bundles over a $4k$-sphere. The following result is best possible for the stated spheres:

THEOREM 1. *A nontrivial stable real vector bundle over S^{4k} is the sum of an irreducible $(2k+1)$-plane bundle and a trivial bundle, if $k>4$.*

This theorem follows from, and implies, the following theorem. The homotopy group $\pi_q(O(n))$ is *stable* for $q<n-1$ (in which case it has been described by Bott [1]), and *metastable* for $q<2(n-1)$. Except for the special cases $n\leqq 12$ the metastable groups are related to the stable groups by

THEOREM 2. *For $q<2(n-1)$ and $n\geqq 13$,*

$$\pi_q(O(n)) = \pi_q(O) \oplus \pi_{q+1}(V_{2n,n}).$$

In fact, splitting occurs in the homotopy sequence of the fibration $O(2n)\rightarrow V_{2n,n}$ at the stated groups. The behaviour in the omitted cases is easily determined from known results.

It follows that the metastable homotopy groups of $O(n)$ exhibit a periodicity, for the second summand is a stable homotopy group of the Stiefel manifold: by [4],

$$\pi_{q+1}(V_{2n,n}) \approx \pi_{q+1}(RP^\infty/RP^{n-1}).$$

Now James has shown [2] that these have a periodicity in a natural way, and in particular that if t denotes the number of nonzero homotopy groups of O in dimensions $\leqq q-n$, then

$$\pi_{q+1}(V_{2n,n}) \approx \pi_{q+1+m-n}(V_{2m,m})$$

for all $m\geqq n$ such that $m-n$ is divisible by 2^t. This isomorphism can be induced by a map of the appropriate skeleton of $V_{2n,n}$ into $\Omega^{m-n}V_{2m,m}$, and so is similar to Bott's periodicity for the stable homotopy groups.

However, the metastable periodicity in $O(n)$ does not arise in exactly the same way as Bott's. The similarity and the difference are shown by the next theorem.